Numerical Analysis I: AMSC/CMSC 666
Homework 4, due Friday, 15 April 2005

(1) Let $(\cdot \mid \cdot)$ be an inner product over $\mathbb{C}^N$. The associated vector norm of $y \in \mathbb{C}^N$ is given by

$$\|y\| = \sqrt{(y \mid y)}\,,$$

while the induced matrix norm of $A \in \mathbb{C}^{N\times N}$ is given by

$$\|A\| = \max\left\{ \frac{\|Ay\|}{\|y\|} \,:\, y \in \mathbb{C}^N\,,\, y \neq 0 \right\}.$$

(a) Show that if A is self-adjoint with respect to the inner product that then

$$\|A\| = \rho_{sp}(A) \equiv \max\{|\lambda| \,:\, \lambda \in \mathrm{Sp}(A)\}$$

(b) Show that if A is self-adjoint with respect to the inner product that then

$$\|A\| = \max\left\{ \frac{|(y \mid Ay)|}{\|y\|^2} \,:\, y \in \mathbb{C}^N\,,\, y \neq 0 \right\}.$$

(c) Show that $\|A^*A\| = \|A\|^2$ for every $A \in \mathbb{C}^{N\times N}$, where A^* is the adjoint of A with respect to the inner product.

(2) Consider $A \in \mathbb{R}^{2\times 2}$ written in the form

$$A = \begin{pmatrix} a+b & c-d \\ c+d & a-b \end{pmatrix}\,, \quad \text{where } a,\, b,\, c,\, d \in \mathbb{R}\,.$$

Compute the following explicitly in terms of a, b, c, and d.

(a) $\rho_{\mathrm{Sp}}(A) = \max\{|\lambda| \,:\, \lambda \in \mathrm{Sp}(A)\}$, where $\mathrm{Sp}(A)$ is the spectrum of A.

(b) $\rho_{\mathrm{Num}}(A) = \max\{|\lambda| \,:\, \lambda \in \mathrm{Num}(A)\}$, where $\mathrm{Num}(A)$ is the numerical range of A, which is defined by

$$\mathrm{Num}(A) = \{y^H A y \,:\, y \in \mathbb{C}^2\,,\, y^H y = 1\}\,.$$

(c) $\|A\|_2$, the matrix norm induced by the Euclidean norm over $\mathbb{C}^2$.

(3) Consider $A \in \mathbb{R}^{2\times 2}$ written in the form

$$A = \begin{pmatrix} a+b & c-d \\ c+d & a-b \end{pmatrix}, \quad \text{where } a,\, b,\, c,\, d \in \mathbb{R}\,.$$

(a) Compute $\rho_{\mathrm{Sp}}(G_J)$ as a function of a, b, c, and d. Identify the values of a, b, c, and d for which the Jacobi method converges.

(b) Do the same for the SOR method as a function of a, b, c, d, and ω.

(4) Find examples of matrices A and G such that

$$A > 0\,, \quad \rho_{\mathrm{Sp}}(G) < 1\,, \quad \text{but} \quad \|G\|_A \geq 1\,.$$

(5) Consider the $N \times N$ matrix

$$A = \begin{pmatrix} \frac{2}{\Delta} & -\frac{1}{\Delta} & 0 & \cdots & 0 \\ -\frac{1}{\Delta} & \frac{2}{\Delta} & -\frac{1}{\Delta} & \ddots & \vdots \\ 0 & -\frac{1}{\Delta} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \frac{2}{\Delta} & -\frac{1}{\Delta} \\ 0 & \cdots & 0 & -\frac{1}{\Delta} & \frac{1}{\Delta} \end{pmatrix},$$

where $\Delta > 0$. Give upper bounds for $\rho_{\mathrm{Sp}}(G_J)$.