

Numerical Analysis I: AMSC/CMSC 666
Homework 5, due Wednesday, 4 May 2005

- (1) Let $A, Q \in \mathbb{R}^{N \times N}$ be self-adjoint and positive definite with respect to a real inner product $(\cdot | \cdot)$ over \mathbb{R}^N . Let κ be the condition number of QA . Prove that

$$1 \leq \frac{(p | Ap)(r | A^{-1}r)}{(p | r)^2} \leq \frac{(\kappa + 1)^2}{4\kappa}$$

for every nonzero $p, r \in \mathbb{R}^N$ such that $p = Qr$.

Hint: Lagrange multipliers. Remark: This fact was used in class without proof to obtain bounds on the error of the A -norm for the preconditioned gradient method.

- (2) Consider \mathbb{R}^2 with the Euclidean inner product. Consider using the steepest descent (gradient) method to solve the 2×2 system

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ c \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & \kappa \end{pmatrix},$$

where $\kappa > 1$. For an initial guess $(x_0, y_0)^T$ the initial residual is

$$\begin{pmatrix} r_0 \\ s_0 \end{pmatrix} = \begin{pmatrix} b \\ c \end{pmatrix} - A \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$

and the steepest descent method produces the sequences

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \alpha_n \begin{pmatrix} r_n \\ s_n \end{pmatrix},$$
$$\begin{pmatrix} r_{n+1} \\ s_{n+1} \end{pmatrix} = \begin{pmatrix} r_n \\ s_n \end{pmatrix} - \alpha_n A \begin{pmatrix} r_n \\ s_n \end{pmatrix},$$

where

$$\alpha_n = \frac{r_n^2 + s_n^2}{r_n^2 + \kappa s_n^2}.$$

- (a) Let e_n denote the A -norm of the n^{th} iterate. Show that

$$e_n \leq \left(\frac{\kappa - 1}{\kappa + 1} \right)^n e_0,$$

and that this bound is sharp for some initial guess $(x_0, y_0)^T$.

- (b) Compute the convergence rate for e_n as a function of $\rho = r_0 s_0 / (r_0^2 + s_0^2)$. Show that in general this rate is better than the upper bound given in part (a).
- (c) Show that the iterates (x_n, y_n) all lie on two lines in the xy -plane.

- (d) Does the Euclidean norm of the error always decrease? Give your reasoning.
- (3) Let $A \in \mathbb{R}^{N \times N}$ be self-adjoint and positive definite with respect to a real inner product $(\cdot | \cdot)$ over \mathbb{R}^N . Let Q_1 and Q_2 be two self-adjoint, positive definite preconditioners for the conjugate gradient method such that the condition number of $Q_2 A$ is nine times the condition number of $Q_1 A$.

(a) Show that the function

$$c \mapsto \left(\frac{c-1}{c+1} \right)^c \text{ is increasing over } [1, \infty).$$

Remark: This fact was used in class without proof to compare convergence rates of various methods.

- (b) Which of the preconditioners will generally produce faster convergence? How much faster when both condition numbers are large? Hint: Use part (a).
- (4) Let $K \in \mathbb{R}^{N \times N}$ be self-adjoint with respect to a real inner product $(\cdot | \cdot)$ over \mathbb{R}^N . For every $p^{(0)} \in \mathbb{R}^N$ the Lanczos algorithm constructs a sequence of vectors $p^{(n)}$ in \mathbb{R}^N as follows:

$$\begin{aligned} p^{(1)} &= Kp^{(0)} - \kappa_0 p^{(0)}, \\ p^{(n+1)} &= Kp^{(n)} - \kappa_n p^{(n)} - \mu_n p^{(n-1)} \quad \text{for } n = 1, 2, \dots, \end{aligned}$$

where the coefficients κ_n and μ_n are given by

$$\begin{aligned} \kappa_n &= \frac{(p^{(n)} | Kp^{(n)})}{(p^{(n)} | p^{(n)})} & \text{for } n = 0, 1, \dots, \\ \mu_n &= \frac{(p^{(n)} | p^{(n)})}{(p^{(n-1)} | p^{(n-1)})} & \text{for } n = 1, 2, \dots \end{aligned}$$

The algorithm halts as soon as $p^{(\bar{n}+1)} = 0$ for some \bar{n} .

(a) Show that the vectors $p^{(n)}$ satisfy the spanning relation

$$\mathcal{K}_{n+1}(p^{(0)}, K) = \text{span}\{p^{(0)}, p^{(1)}, \dots, p^{(n)}\}.$$

(b) Show that the vectors $p^{(n)}$ satisfy the affine relation

$$p^{(n)} \in K^n p^{(0)} + \mathcal{K}_n(p^{(0)}, K),$$

(c) Show that the vectors $p^{(n)}$ satisfy the orthogonality relation

$$(p^{(m)} | p^{(n)}) = 0 \quad \text{for every } m < n \leq \bar{n}.$$

- (5) Let $A, Q \in \mathbb{R}^{N \times N}$ be self-adjoint and positive definite with respect to a real inner product $(\cdot | \cdot)$ over \mathbb{R}^N . Suppose you know that

$$\text{Sp}(QA) \subset [\nu_{min}, \nu_{max}] \subset (0, 2).$$

Set $Q_1 = \frac{2}{\nu_{min} + \nu_{max}}Q$ and $Q_2 = 2Q_1 - Q_1AQ_1$.

- (a) Estimate the convergence rates for the linear iterative methods with growth matrices $I - QA$, $I - Q_1A$, and $I - Q_2A$.
- (b) Estimate the convergence rates for the preconditioned gradient and preconditioned conjugate gradient methods with preconditioners Q_1 and Q_2 .
- (c) Compare the estimated convergence rates for the seven iterative methods above. Giving your reasoning, order them from slowest to fastest.