Numerical Analysis I: AMSC/CMSC 666
Homework 6, due Thursday, 12 May 2005

(1) Consider matrices $A \in \mathbb{R}^{2\times 2}$ written in the form

$$A = \begin{pmatrix} a+b & c-d \\ c+d & a-b \end{pmatrix}, \quad \text{where } a,\, b,\, c,\, d \in \mathbb{R}\,.$$

(a) Show that A has real eigenvalues if and only if

$$d^2 \le b^2 + c^2\,.$$

(b) Characterize in terms of a, b, c, and d when A is an orthogonal matrix.

(c) Show that the set of all orthogonally similar matrices consists of a union of circles — one circle if $d = 0$ and two circles if $d \neq 0$. Give parametric formulas for a, b, c, and d that lie on these circles. Show the radius r of these circles satisfies $r^2 = b^2 + c^2$.

(d) Let the mapping $A \mapsto A'$ denote one step of the unshifted QR iteration. In other words, $A' = RQ$ where $A = QR$ is the unique QR factorization of an invertible matrix A. Give this mapping as a function of a, b, c, and d.

(e) Characterize in terms of a, b, c, and d when the unshifted QR iteration applied to A will converge to a diagonal matrix and give the limiting diagonal matrix. (Use the results of the previous two parts.)

(2) Recall that $A \in \mathbb{C}^{N\times N}$ is called *normal* whenever $A^*A = AA^*$. Show that A is normal and invertible if and only if there exists a unitary matrix U and a self-adjoint, positive definite matrix P such that $A = UP = PU$.

(3) Let $A \in \mathbb{R}^{N\times N}$ be invertible. Let $\{A_n\}_{n=0}^{\infty}$ be the sequence of $N \times N$ matrices constructed recursively by the shifted QR iteration: $A_0 = A$, $A_n = Q_nR_n + \sigma_n I$, and $A_{n+1} = R_nQ_n + \sigma_n I$, where $\{\sigma_n\}_{n=0}^{\infty}$ is a given sequence of shifts, every Q_n is orthogonal, and every R_n is upper triangular with positive diagonal entries.

(a) Show that every A_n is normal whenever A is normal.

(b) Show that every A_n is upper Hessenberg whenever A is upper Hessenberg.

(4) Let $H_0 \in \mathbb{R}^{N\times N}$ and $H(t)$ satisfy the isospectral flow initial-value problem

$$\frac{\mathrm{d}H}{\mathrm{d}t} = JH - HJ\,, \qquad H(0) = H_0\,,$$

where $J(t) \in \mathbb{R}^{N\times N}$ such that $J(t)^T = -J(t)$ for every $t \in \mathbb{R}$. Show that if H_0 is normal then so is $H(t)$ for every $t \in \mathbb{R}$.

(5) Let A be the symmetric tridiagonal real matrix

$$A = \begin{pmatrix} a_0 & b_1 & 0 & \cdots & 0 \\ b_1 & a_1 & b_2 & \ddots & \vdots \\ 0 & b_2 & a_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_n \\ 0 & \cdots & 0 & b_n & a_n \end{pmatrix},$$

where every b_m is nonzero. Let $\{p_m(x)\}_{m=0}^{n+1}$ be the sequence of polynomials generated by

$$p_0(x) = 1\,, \qquad p_1(x) = (x - a_0)\,,$$

$$p_{m+1}(x) = (x - a_m)p_m(x) - b_m^2 p_{m-1}(x) \quad \text{for } m = 1, \cdots, n\,.$$

Let $\pi_0 = 1$, and $\pi_m = b_m \pi_{m-1}$ for every $m = 1, \cdots, n$. Let $q_m(x) = p_m(x)/\pi_m$ for every $m = 0, \cdots, n$. Show that $V^{-1}AV$ is diagonal where

$$V = \begin{pmatrix} q_0(x_0) & q_0(x_1) & q_0(x_2) & \cdots & q_0(x_n) \\ q_1(x_0) & q_1(x_1) & q_1(x_2) & \cdots & q_1(x_n) \\ q_2(x_0) & q_2(x_1) & q_2(x_2) & \cdots & q_2(x_n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_n(x_0) & q_n(x_1) & q_n(x_2) & \cdots & q_n(x_n) \end{pmatrix},$$

and $\{x_k\}_{k=0}^{n+1}$ are the $n+1$ simple roots of $p_{n+1}(x)$.