) If A is self-adjoint, A^*A is positive definite, and we have

$$A^*A u_j = \lambda_j^2 u_j, \quad j=1,\cdots,n, \quad \|u_j\| = 1,$$

let $x = \sum_{j=1}^{n} \alpha_j u_j$,

$$\|Ax\|^2 = (Ax, Ax) = (x, A^*Ax) = \sum_{j=1}^{n} \lambda_j^2 |\alpha_j|^2$$

$$\leq \rho(A^*A) \|x\|^2$$

so, $\|A\|^2 \leq \rho(A^*A)$

Also, for any j, $\lambda_j^2 u_j = A^*A u_j$

$$\|A\|^2 = \left(\max_{\|x\|\neq 0} \frac{\|Ax\|}{\|x\|}\right)^2 = \left(\max_{\|x\|=1} \|Ax\|\right)^2$$

$$\geq \|Au_j\|^2 = (u_j, A^*Au_j) = \lambda_j^2 \leq \rho(A^*A)$$

Thus, $\|A\| = \rho_{sp}(A)$

) $$\max \frac{|(y|Ay)|}{\|y\|^2} \geq \frac{|(u_j|Au_j)|}{\|u_j\|^2} = \lambda_j \quad \forall j=1,\cdots,n$$

so, $$\max \frac{|(y|Ay)|}{\|y\|^2} \geq \rho(A)$$

Also, by Cauchy-Schwarz theorem

$$\frac{|(y|Ay)|}{\|y\|^2} \leq \frac{\|y\|\|Ay\|}{\|y\|^2} = \frac{\|Ay\|}{\|y\|}$$

so, $$\max \frac{|(y|Ay)|}{\|y\|^2} \leq \rho(A)$$ $\square$ hence, proved.

$$\|A^*A\| = \max\left\{\frac{|(y|A^*Ay)|}{\|y\|^2} : y \neq 0\right\}$$

$$= \max\left\{\frac{|(Ay|Ay)|}{\|y\|^2} : y \neq 0\right\}$$

$$= \max\left\{\frac{\|Ay\|^2}{\|y\|^2},\ y \neq 0\right\}$$

$$= \left[\max\left\{\frac{\|Ay\|}{\|y\|}\right\}\right]^2 = \|A\|^2$$

#

2. a)

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda-a-b & d-c \\ -c-d & \lambda-a+b \end{pmatrix}$$

$$= (\lambda-a)^2 + d^2 - b^2 - c^2 = 0 .$$

$\Rightarrow$

$$\rho(A) = \begin{cases} \max\{ a \pm \sqrt{b^2+c^2-d^2} \}, & \text{if } b^2+c^2-d^2 \geq 0 \\ \sqrt{a^2+b^2+c^2-d^2}, & \text{if } b^2+c^2-d^2 < 0 \end{cases}$$

b) Fact: $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$

Let $y = X^T A X$, $X_1 = \cos\theta$, $X_2 = \sin\theta$, $\|X\| = 1$

$$y(x) = \frac{a_{11}+a_{22}}{2} + \frac{a_{11}-a_{22}}{2}\cos 2\theta + \frac{a_{12}+a_{21}}{2}\sin 2\theta$$

$$\max_{\|X\|=1} |X^T A X| = \max\left\{ \frac{a_{11}+a_{22}}{2} \pm \sqrt{\left(\frac{a_{11}-a_{22}}{2}\right)^2 + \left(\frac{a_{12}+a_{21}}{2}\right)^2} \right\}$$

$$= \max\left\{ a \pm \sqrt{b^2+c^2} \right\}$$

c) $$\|A\|_2 = \max_{\|X\|_2=1} \|AX\|_2 = \max_{\|X\|=1} \sqrt{(X^T A^T A X)}$$

$$A^T A = \begin{pmatrix} (a+b)^2 + (c+d)^2 & (a+b)(c-d) + (c+d)(a-b) \\ (a+b)(c-d) + (c+d)(a-b) & (c-d)^2 + (a-b)^2 \end{pmatrix}$$

$$\|A\|_2 = \max\left\{ \left(a^2+b^2+c^2+d^2 \pm 2\sqrt{(a^2+c^2)(b^2+d^2)}\right)^{1/2} \right\}$$

. a)

$$D = \begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix}, \quad W = \begin{pmatrix} 0 & -(c-d) \\ -(c+d) & 0 \end{pmatrix}.$$

$$G_J = D^{-1} W = \begin{pmatrix} 0 & -\frac{c-d}{a+b} \\ -\frac{c+d}{a-b} & 0 \end{pmatrix}$$

$$\det(\lambda I - G_J) = 0$$

$$\Rightarrow \quad \lambda = \pm\sqrt{\frac{c^2-d^2}{a^2-b^2}},$$

so, we need

$$|\lambda| = \sqrt{\frac{c^2-d^2}{a^2-b^2}} < 1 \Rightarrow \left|\frac{c^2-d^2}{a^2-b^2}\right| < 1$$

b)

$$A = D - L - U, \quad D = \begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix}$$

$$L = \begin{pmatrix} 0 & 0 \\ -(c+d) & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & -(c-d) \\ 0 & 0 \end{pmatrix}$$

$$G(w) = (D - wL)^{-1}\left((1-w)D + wU\right)$$

$$\det(\lambda I - G) = 0 \Rightarrow \begin{vmatrix} \lambda - (1-w) & \frac{w(c-d)}{a+b} \\ \frac{w(1-w)(c+d)}{a-b} & \lambda - (1-w) - \frac{w^2(c^2-d^2)}{a^2-b^2} \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda_{1,2} = (1-w) + \frac{w^2(c^2-d^2)}{2(a^2-b^2)} \pm \frac{1}{2}\sqrt{\frac{w^4(c^2-d^2)^2}{(a^2-b^2)^2} + \frac{4(1-w)w^2(c^2-d}{a^2-b^2}}$$

$|\lambda_{max}| = \max\{\lambda_1, \lambda_2\}$, so, we need

$$\max\{|\lambda_1|, |\lambda_2|\} < 1$$

#

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A > 0$,

$G = \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & 1 \end{pmatrix}$, $\rho(G) = \frac{1}{2} < 1$.

$\| G \|_A = \| G \|_2 = \sqrt{\rho(G^T G)}$

$G^T G = \begin{pmatrix} \frac{5}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix} \Rightarrow \rho(G^T G) = \frac{3}{4} + \frac{\sqrt{2}}{2} \geq 1$

$\Rightarrow \| G \|_A \geq 1$

Please check your lecture notes for solution.