1. $p = Qr$

$$\frac{(p\,|\,Ar)(r\,|\,A^{-1}r)}{(p\,|\,r)^2} = \frac{(Qr\,|\,AQr)\,(r\,|\,Q(AQ)^{-1}r)}{(r\,|\,Qr)^2}$$

So, we're looking at the problem:

$$\max_{\|r\|=1} (r\,|\,Ar)(r\,|\,A^{-1}r)\ [\&\ (\min)]$$

under the inner product $(\cdot|\cdot)_Q$,

Apply Lagrange Multipliers method

$$\begin{cases} \nabla_r (r\,|\,Ar)(r\,|\,A^{-1}r) - \lambda \nabla_r[(r\,|\,r)-1] = 0 \\ (r\,|\,r) = 1 \end{cases} \quad (*)$$

where $(\cdot|\cdot)$ are inner product $(\cdot|\cdot)_Q$ and we denote AQ by A.

Solve $(*)$

$$\begin{cases} \lambda = (r\,|\,Ar)(r\,|\,A^{-1}r) \\ A^2 r - 2(r\,|\,Ar)Ar + \dfrac{(r\,|\,Ar)}{(r\,|\,A^{-1}r)}\, r = 0 \end{cases}$$

let $f(A) = A^2 + 2(r\,|\,Ar)A + \dfrac{(r\,|\,Ar)}{(r\,|\,A^{-1}r)}$

Apply Spectral mapping Thm,

for $\lambda_0, \lambda_1 \in Sp(A)$

$$f(\lambda) = (\lambda - \lambda_0)(\lambda - \lambda_1)$$

and $(r|Ar) = \frac{1}{2}(\lambda_0 + \lambda_1)$, $\frac{(r|Ar)}{(r|A^{-1}r)} = \lambda_0 \lambda_1$,

we can solve $(r|A^{-1}r) = \frac{1}{2}\left(\frac{1}{\lambda_0} + \frac{1}{\lambda_1}\right)$

$$(r|Ar)(r|A^{-1}r) = \frac{1}{4}\left(\frac{\lambda_1}{\lambda_0} + \frac{\lambda_0}{\lambda_1} + 2\right)$$

Recall $f(x) = x + \frac{1}{x}$

$$(r|Ar)(r|A^{-1}r) \leq \frac{1}{4}\left(k + \frac{1}{k} + 2\right) = \frac{(1+k)^2}{4k}$$

where $k = \frac{\lambda_{max}}{\lambda_{min}}$, $k = \text{cond}(AQ)$

For Lower bound, look at $q(\lambda)$

since we have real solutions

$$\Delta = 4(r|Ar)^2 - 4\frac{(r|Ar)}{(r|A^{-1}r)} \geq 0$$

$$\Rightarrow \quad (r|Ar)(r|A^{-1}r) \geq 1$$

Thus, we complete the proof.

#

2. $$e_1^2 = r_1^2 + \frac{1}{k} s_1^2 = \left(\frac{k-1}{k+1}\right)^2 e_0^2$$

so, it's sharp.

b) $$\frac{\|e_{n+1}\|_A^2}{\|e_n\|_A^2} = 1 - \frac{(\vec{r}_n | \vec{r}_n)^2}{(\vec{r}_n | A\vec{r}_n)(\vec{r}_n | A^{-1}\vec{r}_n)}$$

$$= 1 - \frac{(r_n^2 + s_n^2)^2}{r_n^4 + s_n^4 + s_n^2 r_n^2 (k + \frac{1}{k})}$$

$$= \frac{s_n^2 r_n^2 (k + \frac{1}{k} - 2)}{r_n^4 + s_n^4 + s_n^2 r_n^2 (k + \frac{1}{k})}$$

$$= \frac{(k-1)^2 \rho_n^2}{k + (k-1)^2 \rho_n^2} \quad \text{where } \rho_n = \frac{r_n s_n}{s_n^2 + r_n^2}$$

Since
$$r_{n+1} = r_n (1 - \alpha_n)$$
$$s_{n+1} = s_n (1 - k\alpha_n),$$
$$\alpha_n = \frac{s_n^2 + r_n^2}{r_n^2 + k s_n^2}$$

$$\rho_{n+1} = \frac{r_n s_n (1-\alpha_n)(1-k\alpha_n)}{s_n^2 (1-k\alpha_n)^2 + r_n^2 (1-\alpha_n)^2} = - \frac{r_n s_n \frac{(k-1) s_n^2}{r_n^2 + k s_n^2} \cdot \frac{(1-k) r_n^2}{r_n^2 + k s_n^2}}{\frac{r_n^2 (k-1)^2 s_n^4}{(r_n^2 + k s_n^2)^2} + \frac{s_n^2 (1-k)^2 r_n^4}{(r_n^2 + s_n^2)^2}}$$

$$= - \frac{r_n s_n}{r_n^2 + s_n^2} = - \rho_n$$

Thus $$\rho_n^2 = \rho_0^2 = \left(\frac{r_0 s_0}{r_0^2 + s_0^2}\right)^2$$

so, $$\frac{\|e_{n+1}\|_A^2}{\|e_n\|_A^2} = \frac{(k-1)^2 \rho^2}{k + (k-1)^2 \rho^2}.$$

2. a) For CG method, we know that

$$e_n \le \left(\frac{\text{cond}\,\#-1}{\text{cond}\,\#+1}\right)^n e_0$$

Let $Q = I$, $\quad \text{cond}(QA) = K$

Since $$\frac{\|e_{n+1}\|_A^2}{\|e_n\|_A^2} = 1 - \frac{(\vec{r}_n \mid \vec{r}_n)^2}{(\vec{r}_n \mid A\vec{r}_n)(\vec{r}_n \mid A^{-1}\vec{r}_n)}$$

where $\vec{r}_n = \begin{pmatrix} r_n \\ s_n \end{pmatrix}$

Apply the result of Q1

$$\frac{\|e_{n+1}\|_A^2}{\|e_n\|_A^2} \le 1 - \frac{4K}{(1+K)^2} = \left(\frac{K-1}{K+1}\right)^2$$

where $K = \dfrac{\lambda_{max}}{\lambda_{min}}$

Thus, we have $\|e_n\|_A \le \left(\frac{K-1}{K+1}\right)^n \|e_0\|_A$

For sharp boundary

Let $X_0 = \begin{pmatrix} b-1 \\ (c-1)/K \end{pmatrix}$

$$\begin{pmatrix} r_0 \\ s_0 \end{pmatrix} = \begin{pmatrix} b \\ c \end{pmatrix} - \begin{pmatrix} 1 & \\ & K \end{pmatrix}\begin{pmatrix} b-1 \\ (c-1)/K \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Then $$\begin{pmatrix} r_1 \\ s_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{2}{1+K}\begin{pmatrix} 1 \\ K \end{pmatrix}$$

2. b)

$$(*) \quad \frac{(k-1)^2 \rho^2}{k+(k-1)^2\rho^2} = \frac{(k-1)^2}{\sqrt{(k+1)^2 + k(\frac{1}{\rho^2}-4)}} ,$$

$$\rho^2 = \left(\frac{r_0 s_0}{r_0^2+s_0^2}\right)^2 \leq \frac{1}{4}$$

which means $(*) \leq \left(\frac{k-1}{k+1}\right)^2$ the upper bound.

2. c) let

$$r_{n+1} = r_n - \frac{r_n^2+s_n^2}{r_n^2+ks_n^2} r_n = \frac{(k-1)s_n^2 r_n}{r_n^2+ks_n^2}$$

$$s_{n+1} = s_n - \frac{r_n^2+s_n^2}{r_n^2+ks_n^2} k s_n = -\frac{(k-1)r_n^2 s_n}{r_n^2+ks_n^2}$$

so,

$$\frac{r_{n+1}}{s_{n+1}} = -\frac{s_n}{r_n}$$

Also,

$$\frac{y_{n+1}-y_n}{x_{n+1}-x_n} = \frac{\alpha_n s_n}{\alpha_n r_n} = \frac{s_n}{r_n}$$

$$\frac{y_{n+2}-y_{n+1}}{x_{n+2}-x_{n+1}} = \frac{\alpha_{n+1} s_{n+1}}{\alpha_{n+1} r_{n+1}} = -\frac{r_n}{s_n}$$

so, (x_n, y_n) actually lie on a perpendicular lines in the xy-plane

2. d) It's always decreasing under A-norm, but not always decreasing under Euclidean norm.

$$\|e_{n+1}\|_{L^2}^2 = r_{n+1}^2 + s_{n+1}^2 / K^2$$

$$\|e_{n+1}\|_{L^2}^2 = \left\| \begin{pmatrix} x - x_n - \alpha_n r_n \\ y - y_n - \alpha_n s_n \end{pmatrix} \right\|_{L^2}^2$$

$$= (x-x_n)^2 - 2\alpha_n r_n (x-x_n) + \alpha_n^2 r_n^2$$

$$+ (y-y_n)^2 - 2\alpha_n s_n (y-y_n) + \alpha_n^2 s_n^2$$

$$= e_n^2 + \frac{(r_n^2+s_n^2)^3 - 2(r_n^2+s_n^2)(r_n^2+\frac{1}{k}s_n^2)}{r_n^2+Ks_n^2}$$

If the last term is positive, then the error is increasing.

let $r_n = 0$, $s_n = \sqrt{\frac{2}{k}} + \varepsilon$,

$$\text{Last term} = (r_n^2+s_n^2)^2 - 2(r_n^2 + \frac{1}{k} s_n^2)$$

$$= \left(s_n^4 - \frac{2s_n^2}{k}\right) = s_n^2\left(s_n^2 - \frac{2}{k}\right) > 0$$

Thus, claim is proved.

3.

a) $f(c) = \left(\frac{c-1}{c+1}\right)^c$

$f'(c) = \left(\frac{c-1}{c+1}\right)^c \left(\ln\frac{c-1}{c+1} + \frac{2c}{c^2-1}\right)$

Let $g = \ln\frac{c-1}{c+1} + \frac{2c}{c^2-1}$

$g'(c) = \frac{-4}{(c^2-1)^2} \le 0$

since $g(+\infty) \ge 0$

$f'(c)$ is positive on $[1, \infty)$

b) $r_2 = \frac{\sqrt{k_2}-1}{\sqrt{k_2}+1} = \frac{3\sqrt{k_1}-1}{3\sqrt{k_1}+1}$, $k_2 = 9k_1$

$r_1 = \frac{\sqrt{k_1}-1}{\sqrt{k_1}+1}$

$\Rightarrow \left(\frac{\sqrt{k_1}-1}{\sqrt{k_1}+1}\right)^{\sqrt{k_1}} \le \left(\frac{\sqrt{k_2}-1}{\sqrt{k_2}+1}\right)^{\sqrt{k_2}}$ (by Part a))

$\Rightarrow \frac{\sqrt{k_1}-1}{\sqrt{k_1}+1} \le \left(\frac{\sqrt{k_2}-1}{\sqrt{k_2}+1}\right)^3$

Method 1 is 3 times faster than method 2.

#

4\. a) $n=1$ is trivial,

Assume $n-1$ is true

$\text{span}\{p^{(0)}, Kp^{(0)}, \dots, K^{n-1}p^{(0)}\} = \text{span}\{p^{(0)}, \dots, p^{(n-1)}\}$

$$p^{(n)} = K p^{(n-1)} - \kappa_{n-1} p^{(n-1)} - \mu_{n-1} p^{(n-2)}$$

$$p^{(n-1)}, p^{(n-2)} \in K_{n-1}(p^{(0)}, K)$$

So,

$$\text{span}\{p^{(0)}, \dots, p^{(n-1)}\} \subseteq \text{span}\{p^{(0)}, \dots, K^n p^{(0)}\}$$

Also,

$$K^n p^{(0)} = K(K^{(n-1)} p^{(0)}) = K^{n-1}(-p^{(1)} + \kappa_0 p^{(0)})$$

$$= K^{n-2}(-Kp^{(1)} + \kappa_0 K p^{(0)})$$

$$= \cdots$$

$$= c\, p^{(n)} + q, \quad \text{where } c \in \mathbb{R}$$

$\in \text{span}\{p^{(0)}, \dots, p^{(n)}\}, q \in \text{span}\{p^{(0)}, \dots, p^{(n-1)}\}$

Thus, we complete the proof.

b) $n=1$ is trivial

Apply the fact

$$p^{(n)} = Kp^{(n-1)} - \kappa_{n-1} p^{(n-1)} - \mu_{n-1} p^{(n-2)}$$

along w/ induction you can prove it.

c) $n=1$ is trivial,

$$(p^{(n+1)} | p^{(n)}) = (Kp^{(n)} - k_n p^{(n)} - \mu_n p^{(n-1)} | p^{(n)})$$

$$= (Kp^n | p^{(n)}) - k_n (p^{(n)} | p^{(n)}) - \mu_n (p^{(n)} | p^{(n-1)})$$

$$= 0$$

And you still need to show

$(p^{(m)} | p^{(n)}) = 0$ when $m = n$, $m < n-1$,

by mimicing the above proof. #

5. Let λ be e-value of QA, $1-\lambda$ is e-value of $I-QA$

a) for $I-QA$

$$\rho(I-QA) = \max[\, v_{max}-1,\ 1-v_{min}\,]$$

for $I-Q_1A$

$$\rho(I-Q_1A) = \frac{v_{max}-v_{min}}{v_{max}+v_{min}}$$

for $I-Q_2A$

$$\rho(I-Q_2A) = 1-\frac{2v_{min}\cdot 2v_{max}}{(v_{min}+v_{max})^2}$$

$$= \frac{(k-1)^2}{(k+1)^2} \quad \text{where } k = \frac{v_{max}}{v_{min}}$$

b) Convergence Rate for Gradient & CG methods:

$$\frac{k-1}{k+1} \quad \& \quad \frac{\sqrt{k}-1}{\sqrt{k}+1}$$

For Q_1A,

$$G_r = \frac{v_{max}-v_{min}}{v_{max}+v_{min}}, \qquad CG_r = \frac{\sqrt{v_{max}}-\sqrt{v_{min}}}{\sqrt{v_{max}}+\sqrt{v_{min}}}$$

For Q_2A,

$$G_r = \frac{(k-1)^2}{(k+1)^2+4k}, \qquad CG_r = \frac{(\sqrt{k}-1)^2}{(\sqrt{k}+1)^2}$$

c) So we have the following order,

$$I - QA < I - Q_1A < I - Q_2A$$

$$G(Q_1) < CG(Q_1)$$

$$G(Q_1) < G(Q_2), \quad CG(Q_1) < CG(Q_2)$$