Let

$$\Pi = \begin{bmatrix} \Pi_0 & & 0 \\ & \Pi_1 & \\ 0 & & \ddots & \\ & & & \Pi_n \end{bmatrix}, \qquad A = \Pi^{-1} \tilde{A} \, \Pi$$

let

$$\begin{bmatrix} p_0(x) \\ \vdots \\ p_n(x) \end{bmatrix} = \Pi \begin{bmatrix} q_0(x) \\ \vdots \\ q_n(x) \end{bmatrix}, \qquad \text{by } (*)$$

$$\tilde{A} \, \Pi \cdot \begin{bmatrix} q_0(x_i) \\ \vdots \\ q_n(x_i) \end{bmatrix} = x_i \, \Pi \begin{bmatrix} q_0(x_i) \\ \vdots \\ q_n(x_i) \end{bmatrix}$$

$$\Rightarrow \quad A \begin{bmatrix} q_0(x_i) \\ \vdots \\ q_n(x_i) \end{bmatrix} = x_i \begin{bmatrix} q_0(x_i) \\ \vdots \\ q_n(x_i) \end{bmatrix}$$

So, x_i are e-values of A, $[q_0(x_i), \cdots, q_n(x_i)]^T$ are corresponding e-vectors of A.

Let $v_i = [q_0(x_i), \cdots, q_n(x_i)]^T$, $V = [v_1, \cdots, v_n]$

we have $A v_i = x_i v_i$

So,

$$V^{-1} A V = \begin{pmatrix} x_0 & & \\ & \ddots & \\ & & x_n \end{pmatrix}$$

#.

1. a) $\det(A - \lambda I) = 0$

$$(a+b-\lambda)(a-b-\lambda) - (c^2-d^2) = 0$$

$$\lambda^2 - 2a\lambda + a^2+d^2-b^2-c^2=0$$

So, A has real e-values iff $\Delta \geq 0$

$$\Delta = 4a^2 - 4(a^2+d^2-b^2-c^2) \geq 0$$

$$\Rightarrow \quad b^2+c^2 \geq d^2$$

b) A is orthogonal $\Rightarrow AA^T = A^TA = I$

$$A \cdot A^T = \begin{bmatrix} (a+b)^2+(c-d)^2 & 2(ac+bd) \\ 2(ac+bd) & (c+d)^2+(a-b)^2 \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A^TA = \begin{bmatrix} (a+b)^2+(c+d)^2 & 2(ac-bd) \\ 2(ac-bd) & (c-d)^2+(a-b)^2 \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad ac = bd = ab = cd = 0 \quad \& \quad a^2+b^2+c^2+d^2 = 1$$

$$\Rightarrow \quad a=0,\ d=0,\ b^2+c^2=1 \quad \text{or} \quad b=0,\ c=0,\ a^2+d^2=1$$

c) Let B and A be orthoganally similar

$$B = UAU^T$$

We have 2 forms of orthoganal matrices

① rotation:

Let $\sin\alpha = \dfrac{b}{\sqrt{b^2+c^2}}$, $\cos\alpha = \dfrac{c}{\sqrt{b^2+c^2}}$, $r = \sqrt{b^2+c^2}$

$$B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a+b & c-d \\ c+d & a-b \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} a + r\left(\frac{b}{r}\cos 2\theta - \frac{c}{r}\sin 2\theta\right) & -d + r\left(\frac{b}{r}\sin 2\theta + \frac{c}{r}\cos 2\theta\right) \\ d + r\left(\frac{b}{r}\sin 2\theta + \frac{c}{r}\cos 2\theta\right) & a - r\left(\frac{b}{r}\cos 2\theta - \frac{c}{r}\sin 2\theta\right) \end{bmatrix}$$

$$= \begin{bmatrix} a + r\sin(\alpha - 2\theta) & r\cos(\alpha - 2\theta) - d \\ r\cos(\alpha - 2\theta) + d & a - r\sin(\alpha - 2\theta) \end{bmatrix}$$

so, a, b, c, d lie on the circles with radius $r^2 = b^2 + c^2$, centered $(a, -d, d, a)$

② Reflection:

$$B = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} a+b & c-d \\ c+d & a-b \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} a + r\sin(\alpha + 2\theta) & -r\cos(\alpha + 2\theta) + d \\ -r\cos(\alpha + 2\theta) - d & a - r\sin(\alpha + 2\theta) \end{bmatrix}$$

so, a, b, c, d lie on the circles w/ radius $r = b^2 + c^2$, centered $(a, d, -d, a)$

If $d = 0$, the two circles are the same.

d) $A' = RQ$, $A = QR$

$A' = RAR^{-1}$, so we need to find R.

Since $R = Q^{-1}A$

$$R^T R = A^T (Q^{-1})^T Q^{-1} A = A^T A$$

$$R = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}, \quad R^T R = \begin{bmatrix} r_{11}^2 & r_{11} r_{12} \\ r_{11} r_{12} & r_{12}^2 + r_{22}^2 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} (a+b)^2 + (c+d)^2 & 2(ac-bd) \\ 2(ac-bd) & (a-b)^2 + (c-d)^2 \end{bmatrix}$$

since $R^T R = A^T A$, we solve R:

$$r_{11} = \sqrt{(a+b)^2 + (c+d)^2}, \quad r_{12} = \frac{2(ac-bd)}{\sqrt{(a+b)^2 + (c+d)^2}}$$

$$r_{22} = \frac{a^2 - b^2 - c^2 + d^2}{\sqrt{(a+b)^2 + (c+d)^2}}$$

so, $A' = R A R^{-1} = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix} \begin{bmatrix} a+b & c-d \\ c+d & a-b \end{bmatrix} \frac{1}{r_{11} r_{12}} \begin{bmatrix} r_{22} & -r_{12} \\ 0 & r_{11} \end{bmatrix}$

e) Apply result c)

A converges to $\begin{bmatrix} a+r & 0 \\ 0 & a-r \end{bmatrix}$

when $d = 0$, $\cos(\alpha + 2\theta) = 0$.

#

2. $A = UP = PU$, $P^* = P$, $U^*U = I = UU^*$

① $AA^* = PUU^*P^* = PP^* = P^2$

$A^*A = P^*U^*UP = P^*P = P^2$

$\Rightarrow AA^* = A^*A$

since U, P are invertible, A is also invertible.

② If A is normal & invertible.

then we have

$$A = VDV^*, \quad \text{where } V \in \mathbb{C}^{N\times N}, \text{ unitary}$$

$$D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}, \quad \lambda_i \text{ are e-value of } A.$$

we can rewrite $D = |D|\Sigma$, where

$$|D| = \begin{pmatrix} |\lambda_1| & & 0 \\ & \ddots & \\ 0 & & |\lambda_N| \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_N \end{pmatrix}$$

where $\sigma_i = \lambda_i / |\lambda_i|$

Now, let $P = V|D| \cdot V^*$, $U = V\Sigma V^*$,

Thus, we find $PU = UP = A$.

#

3. a) For shifted QR method:

$$(A_n - \sigma_n I) Q_n = Q_n R_n Q_n = Q_n (A_{n+1} - \sigma_n I)$$

$$\Rightarrow \quad Q_n^T A_n Q_n = A_{n+1}$$

Let $A_0 = A$, A_0 is normal,

$$A_1 = Q_0^T A_0 Q_0$$

$$A_1 A_1^T = Q_0^T A_0 A_0^T Q_0 = Q_0^T A_0^T A_0 Q_0 = A_1^T A_1$$

so, it's true for $n = 1$,

Suppose A_n is normal,

$$A_{n+1} A_{n+1}^T = (Q_n^T A_n Q_n)(Q_n^T A_n^T Q_n)$$

$$= Q_n^T A_n A_n^T Q_n$$

$$= (Q_n^T A_n^T Q_n)(Q_n^T A_n Q_n)$$

$$= A_{n+1}^T \cdot A_n$$

So, A_n is normal whenever A is normal.

b) $$A_{n+1} = R_n Q_n + \sigma_n I_n$$

$$A_{n+1} R_n = R_n Q_n R_n + R_n \sigma_n$$

$$= R_n (Q_n R_n + \sigma_n I)$$

$$= R_n A_n$$

$$A_{n+1} = R_n A_n R_n^{-1}$$

Fact: If H is upper Hessenberg, R is upper Triangular, then HR, RH are upper Hessenberg we also know that if R is upper triangular, then R^{-1} is also upper triangular.

thus if A_n is upper Hessenberg, A_{n+1} is also Hessenberg, which can be proved by induction.

#

4. Since $J^T = -J$

$$\frac{dH^T}{dt} = (JH - HJ)^T = H^T J^T - J^T H^T$$

$$= JH^T - H^T J, \qquad H^T(0) = H_0^T$$

Thus, $$\frac{d(H^T H)}{dt} = \frac{dH^T}{dt} H + H^T \frac{dH}{dt}$$

$$= (JH^T - H^T J) H + H^T (JH - HJ)$$

$$= JH^T H - H^T H J$$

$$H^T(0) H(0) = H_0^T H_0$$

$$\frac{d(HH^T)}{dt} = \frac{dH}{dt} H^T + H \frac{dH^T}{dt}$$

$$= (JH - HJ) H^T + H (JH^T - H^T J)$$

$$= JHH^T - HH^T J, \quad H(0) H(0)^T = H_0 H_0^T$$

Since H_0 is normal, HH^T & $H^T H$ are gouened by the same system, so, $HH^T = H^T H$ for all t, which means $H(t)$ is normal.

#

5. Given $\{P_m(x)\}_{m=0}^{n+1}$, $P_0(x)=1$, $P_1(x)=x-a_0$,
$P_{m+1}(x) = (x-a_m)P_m(x) - b_m^2 P_{m-1}(x)$,
we have

$$\begin{bmatrix} x-a_0 & -1 & & \\ -b_1^2 & x-a_1 & -1 & \\ & -b_2^2 & \ddots & \ddots \\ & & \ddots & \ddots & -1 \\ & & & -b_n^2 & x-a_n \end{bmatrix} \begin{bmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ P_n(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ P_{n+1}(x) \end{bmatrix}$$

let $P_{n+1}(x_i)=0$, which means x_i are roots of $P_{n+1}(x)$.

thus,

$$\begin{bmatrix} x_i-a_0 & -1 & & \\ -b_1^2 & x_i-a_1 & -1 & \\ & -b_2^2 & \ddots & \ddots \\ & & \ddots & \ddots & -1 \\ & & & -b_n^2 & x_i-a_n \end{bmatrix} \begin{bmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ P_n(x) \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} a_0 & 1 & & \\ b_1^2 & a_1 & 1 & \\ & b_2^2 & \ddots & \ddots \\ & & \ddots & \ddots & 1 \\ & & & b_n^2 & a_n \end{bmatrix} \begin{bmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ P_n(x) \end{bmatrix} = x_i \begin{bmatrix} P_0(x) \\ \vdots \\ \vdots \\ P_n(x) \end{bmatrix} \quad (*)$$

Denote the tridiagonal matrix as $\tilde{A}$, so x_i are e-values of $\tilde{A}$, $[P_0(x_i), \dots, P_n(x_i)]^T$ are e-vectors of $\tilde{A}$.