Midterm Exam: AMSC/CMSC 666
Thursday, 17 March 2005
Solutions

(1) Let $f(x) = 1 + x^6$. Give the quadratic Chebyschev interpolation of f over $[-1, 1]$. Give a uniform bound on the error of this approximation over $[-1, 1]$.

Solution. Quadratic interpolations are determined by three points. You therefore need the three roots of $T_3(x)$, the third Chebyschev polynomial. By partitioning the upper unit semicircle into three equal arcs, the midpoints are at the angles $-\frac{\pi}{6}$, $\frac{\pi}{2}$, $\frac{\pi}{6}$. The roots of $T_3(x)$ are the cosines of these angles — namely, $-\sqrt{\frac{3}{4}}$, 0, $\sqrt{\frac{3}{4}}$. Alternatively, you might recall that $T_3(x) = 4x^3 - ax$, and that $T_3(1) = 1$, thereby inferring that $T_3(x) = 4x^3 - 3x$.

Because $f(x) = 1 + x^6$ over $[-1, 1]$ is even, its quadratic Chebyschev interpolation will also be even. Indeed, it is clearly given by

$$p(x) = 1 + \tfrac{6}{16}x^2 \,,$$

which clearly satisfies $p(x) = f(x)$ at $x = -\sqrt{\frac{3}{4}}$, 0, $\sqrt{\frac{3}{4}}$.

The error of this interpolation is given by

$$e(x) = p(x) - f(x) = \tfrac{6}{16}x^2 - x^6 \,.$$

An elementary Calculus One argument yields that $|e(x)| \leq -e(1) = \frac{7}{16}$. Of course, any number larger than $\frac{7}{16}$ also bounds the error. $\square$

(2) Derive the one- and two-point Gaussian quadrature formulas such that

$$\int_{-1}^{1} f(x)x^6 \, \mathrm{d}x \approx \sum_{j=1}^{n} f(x_j)\, w_j \,.$$

Give bounds on the error of these formulas.

Solution. It is relatively easy to see that the first three monic orthogonal polynomials are

$$p_0(x) = 1 \,, \qquad p_1(x) = x \,, \qquad p_2(x) = x^2 - \tfrac{7}{9} \,.$$

The one- and two-point Gaussian quadrature formulas therefore have the form

$$Q_1(f) = f(0)w_0 \,, \qquad Q_2(f) = f\Big(-\sqrt{\tfrac{7}{9}}\Big)\, w_- + f\Big(\sqrt{\tfrac{7}{9}}\Big)\, w_+ \,.$$

Because $Q_1(1) = \frac{2}{7}$, we see that $w_0 = \frac{2}{7}$. By the even symmetry of the x^6 weight, we conclude that $w_- = w_+$. Then because $Q_2(1) = \frac{2}{7}$, we see that $w_- = w_+ = \frac{1}{7}$. Hence, the one- and two-point Gaussian quadrature formulas are

$$Q_1(f) = f(0)\tfrac{2}{7}\,, \qquad Q_2(f) = f\Big(-\sqrt{\tfrac{7}{9}}\Big)\,\tfrac{1}{7} + f\Big(\sqrt{\tfrac{7}{9}}\Big)\,\tfrac{1}{7}\,.$$

The corresponding errors can be bound by

$$\begin{aligned}
|E_1(f)| &\leq \frac{1}{2!}\int_{-1}^{1} |p_1(x)|^2 x^6 \,\mathrm{d}x \\
&= \frac{1}{2}\int_{-1}^{1} x^8 \,\mathrm{d}x = \frac{1}{9}\,, \\
|E_2(f)| &\leq \frac{1}{4!}\int_{-1}^{1} |p_2(x)|^2 x^6 \,\mathrm{d}x \\
&= \frac{1}{24}\int_{-1}^{1} \big(x^2 - \tfrac{7}{9}\big)^2 x^6 \,\mathrm{d}x \\
&= \frac{1}{24}\int_{-1}^{1} \big(x^2 - \tfrac{7}{9}\big) x^8 \,\mathrm{d}x \\
&= \frac{1}{24}\int_{-1}^{1} x^{10} - \tfrac{7}{9} x^8 \,\mathrm{d}x \\
&= \frac{2}{24}\left[\frac{1}{11} - \frac{7}{81}\right] = \frac{1}{12}\,\frac{81-77}{11\cdot 81} = \frac{1}{2673}\,.
\end{aligned}$$

□

(3) Partition the interval $[x_L, x_R]$ into n subintervals as

$$x_L = x_0 < x_1 < x_2 < \cdots < x_{n-2} < x_{n-1} < x_n = x_R\,.$$

Prove that the linear spline minimizes the integral

$$\int_{x_L}^{x_R} |Y'(x)|^2 \,\mathrm{d}x\,,$$

subject to the constraints $Y(x_i) = y_i$ for $i = 0, \cdots, n$ over the class of functions Y that are continuous over $[x_L, x_R]$ and are smooth with bounded derivatives over the subinterval (x_{i-1}, x_i) for every $i = 1, \cdots, n$.

Solution. The linear spline is given by

$$S(x) = \frac{x_i - x}{x_i - x_{i-1}}\, y_{i-1} + \frac{x - x_{i-1}}{x_i - x_{i-1}}\, y_i$$

when $x \in [x_{i-1}, x_i]$ for every $i = 1, \cdots, n\,.$

Let Y be continuous over $[x_L, x_R]$ with bounded derivatives over the subinterval (x_{i-1}, x_i) for every $i = 1, \cdots, n$. Set $Z = Y - S$. Then $Z(x_i) = 0$ for $i = 0, \cdots, n$. Consider the identity

$$\int_{x_L}^{x_R} |Y'(x)|^2 \,\mathrm{d}x = \int_{x_L}^{x_R} |S'(x)|^2 \,\mathrm{d}x + 2\int_{x_L}^{x_R} Z'(x)S'(x)\,\mathrm{d}x + \int_{x_L}^{x_R} |Z'(x)|^2 \,\mathrm{d}x \,.$$

Integration by parts shows that for every $i = 0, \cdots, n$ one has

$$\int_{x_{i-1}}^{x_i} S'(x)Z'(x)\,\mathrm{d}x = S'(x)Z(x)\bigg|_{x_{i-1}}^{x_i} - \int_{x_{i-1}}^{x_i} S(x)Z''(x)\,\mathrm{d}x = 0\,,$$

whereby

$$\int_{x_L}^{x_R} Z'(x)S'(x)\,\mathrm{d}x = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} S'(x)Z'(x)\,\mathrm{d}x = 0\,.$$

We thereby see that

$$\begin{aligned}\int_{x_L}^{x_R} |Y'(x)|^2 \,\mathrm{d}x &= \int_{x_L}^{x_R} |S'(x)|^2 \,\mathrm{d}x + \int_{x_L}^{x_R} |Z'(x)|^2 \,\mathrm{d}x \\ &\geq \int_{x_L}^{x_R} |S'(x)|^2 \,\mathrm{d}x\,.\end{aligned}$$

The linear spline therefore minimizes the integral. □

(4) Let $f \in C^1(\mathbb{S}^1; \mathbb{C})$. Let

$$S^n f(x) = \sum_{k=-n}^{n} \hat{f}_k e^{ikx}\,,$$

where

$$\hat{f}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikx} f(x)\,\mathrm{d}x\,.$$

Show there exists an $M < \infty$ such that

$$\big\|S^n f - f\big\|_2 \leq \frac{M}{n^{\frac{1}{2}}} \quad \text{for every } n \in \mathbb{N}\,.$$

Here the norm $\|\cdot\|_2$ is given by

$$\|g\|_2 = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(x)|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}\,.$$

Solution. Because $f \in C^1(\mathbb{S}^1;\mathbb{C})$ we have that

$$ik\hat{f}_k = \widehat{f'}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikx} f'(x)\,\mathrm{d}x\,.$$

The integral on the right-hand side above can be bounded by

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikx} f'(x)\,\mathrm{d}x\right| \leq \|f'\|_\infty\,,$$

whereby we can bound $\hat{f}_k$ as

$$|\hat{f}_k| \leq \frac{\|f'\|_\infty}{|k|} \quad \text{for } k \neq 0\,.$$

Because $\{e^{ikx}\}_{k\in\mathbb{Z}}$ is orthonormal, we then use Parsval and the above bound on $\hat{f}_k$ to obtain

$$\begin{aligned}\|S^n f - f\|_2^2 &= \Big\|\sum_{|k|>n} \hat{f}_k e^{ikx}\Big\|_2^2 \\ &= \sum_{|k|>n} |\hat{f}_k|^2 \\ &\leq 2\|f'\|_\infty^2 \sum_{k=n+1}^{\infty} \frac{1}{k^2}\,.\end{aligned}$$

Next, you can compare with an integral to argue that

$$\sum_{k=n+1}^{\infty} \frac{1}{k^2} \leq \int_n^\infty \frac{1}{s^2}\,\mathrm{d}s = \frac{1}{n}\,.$$

Alternatively, you can compare with a telescoping sum to argue that

$$\sum_{k=n+1}^{\infty} \frac{1}{k^2} \leq \sum_{k=n+1}^{\infty} \frac{1}{k(k-1)} = \sum_{k=n+1}^{\infty} \left[\frac{1}{k-1} - \frac{1}{k}\right] = \frac{1}{n}\,.$$

In either case, you obtain the bound

$$\|S^n f - f\|_2^2 \leq 2\|f'\|_\infty^2 \frac{1}{n}\,.$$

This yields the desired bound upon taking the square root of both sides and setting $M = \sqrt{2}\|f'\|_\infty$. □

(5) Consider the definite integral

$$I(f) = \int_{x_L}^{x_R} f(x)\,\mathrm{d}x\,.$$

Let $T_\Delta(f)$ and $M_\Delta(f)$ denote the approximations of $I(f)$ by the trapezoidal and midpoint rules with uniform subintervals of length Δ. As $\Delta \to 0$ the Euler-Maclaurin formula states that the trapezoidal rule satisfies the asymptotic relation

$$T_\Delta(f) = I(f) + \alpha_2\Delta^2 + \alpha_4\Delta^4 + O(\Delta^6)\,.$$

(a) Use the fact that $T_{\frac{1}{2}\Delta}(f) = \frac{1}{2}[T_\Delta(f) + M_\Delta(f)]$ to derive a similar asymptotic relation for the midpoint rule. Your answer should be in terms of α_2 and α_4.

(b) Extrapolate $M_\Delta(f)$ and $M_{3\Delta}(f)$ to obtain a fourth order accurate quadrature.

Solution. Because

$$T_\Delta(f) = I(f) + \alpha_2\Delta^2 + \alpha_4\Delta^4 + O(\Delta^6)\,,$$
$$T_{\frac{1}{2}\Delta}(f) = I(f) + \tfrac{1}{4}\alpha_2\Delta^2 + \tfrac{1}{16}\alpha_4\Delta^4 + O(\Delta^6)\,,$$

we see that the answer to part (a) is

$$\begin{aligned} M_\Delta(f) &= 2T_{\frac{1}{2}\Delta}(f) - T_\Delta(f) \\ &= I(f) - \tfrac{1}{2}\alpha_2\Delta^2 - \tfrac{7}{8}\alpha_4\Delta^4 + O(\Delta^6)\,. \end{aligned}$$

It then follows that

$$M_{3\Delta}(f) = I(f) - \tfrac{1}{2}9\alpha_2\Delta^2 - \tfrac{7}{8}81\alpha_4\Delta^4 + O(\Delta^6)\,,$$

whereby one obtains the fourth-order accurate quadrature

$$\begin{aligned} Q_\Delta(f) &= \frac{9M_\Delta(f) - M_{3\Delta}(f)}{8} \\ &= I(f) + \tfrac{7}{8}72\alpha_4\Delta^4 + O(\Delta^6) \\ &= I(f) + 63\alpha_4\Delta^4 + O(\Delta^6)\,. \end{aligned}$$

□