First Exam Solutions: MATH 410
Friday, 29 September 2006

1. [10] Suppose that $a \in \mathbb{R}$ has the property that $a < 1/k$ for every $k \in \mathbb{Z}_+$. Show that $a \leq 0$.

   Solution: Suppose $a \leq 0$ does not hold. Then by trichotomy $a > 0$. By the Archimedean Property there exists $n \in \mathbb{Z}_+$ such that $1 < na$. Then $1/n < a$, which contradicts the property that $a < 1/k$ for every $k \in \mathbb{Z}_+$. Therefore $a \leq 0$ holds. $\square$

   An alternative solution that uses more advanced machinery (and therefore is not as good) is the following. Because constant sequences converge while $1/k \to 0$ as $k \to \infty$, and because of the way limits preserve inequalities, one has

$$a = \lim_{k\to\infty} a \leq \lim_{k\to\infty} 1/k = 0 . \quad \square$$

   Remark: The Archimedean Property lies behind the fact that $1/k \to 0$ as $k \to \infty$ in the alternative solution above.

2. [20] Give a counterexample to each of the following assertions.
   (a) Every monotonic sequence in $\mathbb{R}$ converges.
   (b) Every bounded sequence in $\mathbb{R}$ converges.
   (c) A sequence $\{a_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}$ is convergent if the sequence $\{a_k^2\}_{k\in\mathbb{N}}$ is convergent.
   (d) A countable union of closed subsets of $\mathbb{R}$ is closed.

   Solution (a): Let $a_k = k$ for every $k \in \mathbb{N}$. Then the sequence $\{a_k\}_{k\in\mathbb{N}}$ is increasing (and therefore monotonic), but does not converge. $\square$

   Solution (b): Let $a_k = (-1)^k$ for every $k \in \mathbb{N}$. Then the sequence $\{a_k\}_{k\in\mathbb{N}}$ is bounded ($-1 \leq a_k \leq 1$), but does not converge. $\square$

   Solution (c): Let $a_k = (-1)^k$ for every $k \in \mathbb{N}$. Then the sequence $\{a_k^2\}_{k\in\mathbb{N}}$ converges to 1 (because $a_k^2 = (-1)^{2k} = 1$), while the sequence $\{a_k\}_{k\in\mathbb{N}}$ does not converge. $\square$

   Solution (d): Let $I_k = [\frac{1}{2^k}, 2]$ for every $k \in \mathbb{N}$. Then each interval I_k is closed while

$$\bigcup_{k\in\mathbb{N}} I_k = (0,2] \quad \text{is not closed} . \quad \square$$

3. [20] Consider the real sequence $\{b_k\}_{k\in\mathbb{N}}$ given by

$$b_k = (-1)^k \, \frac{3k+4}{k+1} \qquad \text{for every } k \in \mathbb{N} = \{0, 1, 2, \dots\}\,.$$

(a) Give the first three terms of the subsequence $\{b_{2k+1}\}_{k\in\mathbb{N}}$.
(b) Give the first three terms of the subsequence $\{b_{3^k}\}_{k\in\mathbb{N}}$.
(c) Give $\limsup\limits_{k\to\infty} b_k$ and $\liminf\limits_{k\to\infty} b_k$. (No proof is needed here.)

Solution: You are given that $\mathbb{N} = \{0, 1, 2, \cdots\}$, as was done in class and in the notes (but in not the book). Then (a) the first three terms of the subsequence $\{b_{2k+1}\}_{k\in\mathbb{N}}$ are

$$b_1 = -\tfrac{7}{2}\,, \qquad b_3 = -\tfrac{13}{4}\,, \qquad b_5 = -\tfrac{19}{6}\,,$$

while (b) the first three terms of the subsequence $\{b_{3^k}\}_{k\in\mathbb{N}}$ are

$$b_1 = -\tfrac{7}{2}\,, \qquad b_3 = -\tfrac{13}{4}\,, \qquad b_9 = -\tfrac{31}{10}\,.$$

Because $b_{2k+1} < 0$ while

$$\lim_{k\to\infty} b_{2k} = \lim_{k\to\infty} \frac{6k+4}{2k+1} = 3\,,$$

and because $b_{2k} > 0$ while

$$\lim_{k\to\infty} b_{2k+1} = -\lim_{k\to\infty} \frac{6k+7}{2k+2} = -3\,,$$

(c) one has respectively that

$$\limsup_{k\to\infty} b_k = 3\,, \quad \text{and} \quad \liminf_{k\to\infty} b_k = -3\,.$$

□

4. [20] Let $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ be bounded sequences in $\mathbb{R}$.
(a) Prove that

$$\limsup_{k\to\infty}\,(a_k + b_k) \le \limsup_{k\to\infty} a_k + \limsup_{k\to\infty} b_k\,.$$

(b) Give an example for which equality does not hold above.

Solution (a): Let $c_k = a_k + b_k$ for every $k \in \mathbb{N}$. Because $\{a_k\}_{k\in\mathbb{N}}$, $\{b_k\}_{k\in\mathbb{N}}$, and $\{c_k\}_{k\in\mathbb{N}}$ are bounded sequences in $\mathbb{R}$, we have

$$\begin{aligned}
\limsup_{k\to\infty} a_k &= \lim_{k\to\infty} \overline{a}_k\,,\\
\limsup_{k\to\infty} b_k &= \lim_{k\to\infty} \overline{b}_k\,,\\
\limsup_{k\to\infty} c_k &= \lim_{k\to\infty} \overline{c}_k\,,
\end{aligned}$$

where $\{\overline{a}_k\}_{k\in\mathbb{N}}$, $\{\overline{b}_k\}_{k\in\mathbb{N}}$, and $\{\overline{c}_k\}_{k\in\mathbb{N}}$ are the bounded, nonincreasing (and therefore convergent) sequences in $\mathbb{R}$ whose terms are defined for every $k \in \mathbb{N}$ by

$$\begin{aligned}\overline{a}_k &= \sup\{a_l \,:\, l \geq k\}\,,\\ \overline{b}_k &= \sup\{b_l \,:\, l \geq k\}\,,\\ \overline{c}_k &= \sup\{c_l \,:\, l \geq k\}\,.\end{aligned}$$

For every $k \in \mathbb{N}$ we have

$$a_l \leq \overline{a}_k \quad \text{and} \quad b_l \leq \overline{b}_k \quad \text{for every } l \geq k\,,$$

whereby

$$c_l = a_l + b_l \leq \overline{a}_k + \overline{b}_k \quad \text{for every } l \geq k\,.$$

By taking the sup over those l with $l \geq k$ above, we see that

$$\overline{c}_k = \sup\{c_l \,:\, l \geq k\} \leq \overline{a}_k + \overline{b}_k\,.$$

Then by the properties of limits

$$\begin{aligned}\limsup_{k\to\infty}\,(a_k + b_k) &= \limsup_{k\to\infty} c_k\\ &= \lim_{k\to\infty} \overline{c}_k\\ &\leq \lim_{k\to\infty} (\overline{a}_k + \overline{b}_k)\\ &= \lim_{k\to\infty} \overline{a}_k + \lim_{k\to\infty} \overline{b}_k\\ &= \limsup_{k\to\infty} a_k + \limsup_{k\to\infty} b_k\,. \quad \square\end{aligned}$$

Solution (b): Let $a_k = (-1)^k$ and $b_k = (-1)^{k+1}$ for every $k \in \mathbb{N}$. Clearly

$$\begin{aligned}\limsup_{k\to\infty} a_k &= \lim_{k\to\infty} a_{2k} = 1\,,\\ \limsup_{k\to\infty} b_k &= \lim_{k\to\infty} b_{2k+1} = 1\,,\end{aligned}$$

while (because $a_k + b_k = 0$ for every $k \in \mathbb{N}$)

$$\limsup_{k\to\infty}\,(a_k + b_k) = \lim_{k\to\infty}\,(a_k + b_k) = 0\,.$$

Therefore

$$\limsup_{k\to\infty}\,(a_k + b_k) = 0 < 2 = \limsup_{k\to\infty} a_k + \limsup_{k\to\infty} b_k\,. \quad \square$$

5. [20] Let A and B be subsets of $\mathbb{R}$.
   (a) Prove that $(A \cap B)^c \subset A^c \cap B^c$.
   (b) Give an example for which equality does not hold above.

Solution (a): Let $x \in (A \cap B)^c$. By the definition of closure, there exists a sequence $\{x_k\}_{k\in\mathbb{N}}$ contained in $A \cap B$ such that $x_k \to x$ as $k \to \infty$. But the sequence $\{x_k\}_{k\in\mathbb{N}}$ is therefore contained in both A and B while $x_k \to x$ as $k \to \infty$. By the definition of closure, it follows that $x \in A^c$ and $x \in B^c$, whereby $x \in A^c \cap B^c$. $\square$

Solution (b): A simple example is $A = (0, 1)$ and $B = (1, 2)$. Then $(A \cap B)^c = \emptyset^c = \emptyset$ (because $A \cap B = \emptyset$), while $A^c \cap B^c = [0, 1] \cap [1, 2] = [1, 1] = \{1\}$ (because $A^c = [0, 1]$ and $B^c = [1, 2]$). Hence, $(A \cap B)^c = \emptyset \neq \{1\} = A^c \cap B^c$. $\square$

A more dramatic example is $A = \mathbb{Q}$ and $B = \{\sqrt{2}+q : q \in \mathbb{Q}\}$. Notice that $A \cap B = \emptyset$ because $\sqrt{2}$ is irrational. Notice too that $A^c = \mathbb{R}$ and $B^c = \mathbb{R}$ — i.e. that A and B are each dense in $\mathbb{R}$. Then $(A \cap B)^c = \emptyset^c = \emptyset$, while $A^c \cap B^c = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$. Hence, $(A \cap B)^c = \emptyset \neq \mathbb{R} = A^c \cap B^c$. $\square$

6. [10] Let $\{b_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ and let A be a subset of $\mathbb{R}$. Write the negations of the following assertions.
   (a) "For every $m \in \mathbb{R}$ one has $b_j > m$ frequently as $j \to \infty$."
   (b) "Every sequence in A has a subsequence that converges to a limit in A."

Solution (a): "There exists $m \in \mathbb{R}$ such that $b_j \leq m$ ultimately as $j \to \infty$." $\square$

Solution (b): "There is a sequence in A such that every subsequence of it will diverge or will converge to a limit outside A." $\square$

Remark: The solution to (b) becomes clearer if you rephrase the original assertion "For every sequence in A there exists a subsequence that will converge and will have its limit in A." The negation of "will converge and will have its limit in A" is "will diverge or will converge to a limit outside A."

$\square$