

First In-Class Exam Solutions
Math 246, Spring 2008, Professor David Levermore

- (1) [12] Suppose you are using numerical methods to approximate the solution of an initial-value problem over the time interval $[0, 5]$. By what factor would you expect the error to decrease if you increase the number of time steps taken from 500 to 1500 when you use each of the following explicit methods with a uniform time step h .

(a) Heun-trapezoidal method

Solution: This method is second order, so its error scales like h^2 . When h decreases by a factor of 3 the error will therefore decrease by a factor of $3^2 = 9$.

(b) Runge-Kutta method

Solution: This method is fourth order, so its error scales like h^4 . When h decreases by a factor of 3 the error will therefore decrease by a factor of $3^4 = 81$.

(c) Euler method

Solution: This method is first order, so its error scales like h . When h decreases by a factor of 3 the error will therefore decrease by a factor of 3.

(d) Heun-midpoint method

Solution: This method is second order, so its error scales like h^2 . When h decreases by a factor of 3 the error will therefore decrease by a factor of $3^2 = 9$.

- (2) [20] Find the explicit solution for each of the following initial-value problems and identify its interval of existence (interval of definition).

(a) $t \frac{dz}{dt} + 4z = 6t^2, \quad z(1) = 4.$

Solution: This equation is linear. Its linear normal form is

$$\frac{dz}{dt} + \frac{4}{t}z = 6t.$$

An integrating factor is $\exp\left(\int_1^t \frac{4}{s} ds\right) = \exp(4 \log(|t|)) = t^4$, so that

$$\frac{d}{dt}(t^4 z) = 6t^5, \quad \implies \quad t^4 z = t^6 + C.$$

The initial condition $z(1) = 4$ implies that $C = 1^4 \cdot 4 - 1^6 = 3$. Therefore

$$z = t^2 + \frac{3}{t^4}, \quad \text{with interval of existence } t > 0.$$

(b) $\frac{du}{dx} = \frac{12x^2}{1+u}, \quad u(0) = 0.$

Solution: This equation is separable. Its separated differential form is

$$(u+1) du = 12x^2 dx, \quad \implies \quad \frac{1}{2}(u+1)^2 = 4x^3 + C.$$

The initial condition $u(0) = 0$ implies that $C = \frac{1}{2}(0+1)^2 - 4 \cdot 0^3 = \frac{1}{2}$. Therefore $(u+1)^2 = 8x^3 + 1$, which can be solved as

$$u = -1 + \sqrt{8x^3 + 1}, \quad \text{with interval of existence } x > -\frac{1}{8}.$$

The positive square root is needed to satisfy the initial condition.

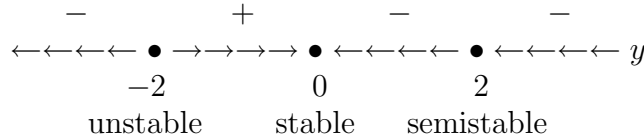
(3) [18] Consider the differential equation $\frac{dy}{dt} = -(y+2)y(y-2)^2$.

(a) Sketch its phase-line. Indicate all of the stationary (equilibrium) solutions and classify each as being either stable, unstable, or semistable.

(b) If $y(0) = -1$, how does the solution $y(t)$ behave as $t \rightarrow \infty$?

(c) If $y(0) = 4$, how does the solution $y(t)$ behave as $t \rightarrow \infty$?

Solution (a): The stationary solutions are $y = -2$, $y = 0$, and $y = 2$. A sign analysis of $-(y+2)y(y-2)^2$ shows that the phase-line for this equation is therefore



(b): The phase-line shows that if $y(0) = -1$ then $y(t) \rightarrow 0$ as $t \rightarrow \infty$.

(c): The phase-line shows that if $y(0) = 4$ then $y(t) \rightarrow 2$ as $t \rightarrow \infty$.

(4) [18] Consider the following MATLAB function M-file.

```
function [t,y] = solveit(ti, yi, tf, n)

h = (tf - ti)/n;
t = zeros(n + 1, 1);
y = zeros(n + 1, 1);
t(1) = ti;
y(1) = yi;
for k = 1:n
t(k + 1) = t(k) + h;
y(k + 1) = y(k) + h*(2*y(k) - (y(k))^2);
end
```

(a) What is the initial-value problem being approximated numerically?

(b) What is the numerical method being used?

(c) What are the output values of $t(2)$ and $y(2)$ that you would expect for input values of $t_i = 0$, $y_i = 1$, $t_f = 5$, $n = 50$?

Solution (a): The initial-value problem being solved is $\frac{dy}{dt} = 2y - y^2$, $y(t_i) = y_i$.

(b): It is being approximated by the (forward, explicit) Euler method.

(c): When $t_i = 0$, $y_i = 1$, $t_f = 5$, $n = 50$ one has $h = (t_f - t_i)/n = (5 - 0)/50 = .1$, $t(1) = t_i = 0$, and $y(1) = y_i = 1$. Hence, $t(2) = t(1) + h = 0 + .1 = .1$, while

$$y(2) = y(1) + h (2 y(1) - y(1)^2) = 1 + .1(2 \cdot 1 - 1^2) = 1.1.$$

- (5) [12] There are 200,000 mosquitoes in a certain area initially. In the absence of predators this population of mosquitoes would increase at a rate proportional to the current population and would triple each week. However, predators eat 140,000 mosquitoes per week at a constant rate.
- (a) Write down an initial-value problem that governs the population of mosquitoes in the area at any positive time.
- (b) Will this population increase or decrease over time?

Solution (a): Let $P(t)$ be the population of mosquitoes at time t weeks. Tripling each week implies a growth rate of $\log(3)$. The initial-value problem satisfied by P is

$$\frac{dP}{dt} = \log(3)P - 140,000, \quad P(0) = 200,000.$$

(b): Because $\frac{dP}{dt} = \log(3)(P - \frac{140,000}{\log(3)}) > 0$ for $P > \frac{140,000}{\log(3)}$, and because $\log(3) > 1$ implies $P(0) = 200,000 > \frac{140,000}{\log(3)}$, the population will increase over time.

- (6) [20] Give an implicit general solution to each of the following differential equations.
- (a) $(xy^2 + y + e^x) dx + (x^2y + x) dy = 0$.

Solution: This differential form is exact because

$$\partial_y(xy^2 + y + e^x) = 2xy + 1 = \partial_x(x^2y + x) = 2xy + 1.$$

We can therefore find $H(x, y)$ such that

$$\partial_x H(x, y) = xy^2 + y + e^x, \quad \partial_y H(x, y) = x^2y + x.$$

The second equation implies $H(x, y) = \frac{1}{2}x^2y^2 + xy + h(x)$. Plugging this into the first equation gives

$$xy^2 + y + h'(x) = \partial_x H(x, y) = xy^2 + y + e^x,$$

which yields $h'(x) = e^x$. Taking $h(x) = e^x$, the general solution is

$$\frac{1}{2}x^2y^2 + xy + e^x = C.$$

- (b) $(2e^x + y^3) dx + 3y^2 dy = 0$.

Solution: This differential form is not exact because

$$\partial_y(2e^x + y^3) = 3y^2 \neq \partial_x(3y^2) = 0.$$

We seek an integrating factor μ such that

$$\partial_y[(2e^x + y^3)\mu] = \partial_x[(3y^2)\mu].$$

This is just $(2e^x + y^3)\partial_y\mu + 3y^2\mu = 3y^2\partial_x\mu$. If $\partial_y\mu = 0$ it reduces to $\mu = \partial_x\mu$, whereby $\mu = e^x$. We can therefore find $H(x, y)$ such that

$$\partial_x H(x, y) = (2e^x + y^3)e^x, \quad \partial_y H(x, y) = 3y^2e^x.$$

The second equation implies $H(x, y) = y^3e^x + h(x)$. Plugging this into the first equation gives

$$y^3e^x + h'(x) = \partial_x H(x, y) = (2e^x + y^3)e^x,$$

which yields $h'(x) = 2e^{2x}$. Taking $h(x) = e^{2x}$, the general solution is

$$y^3e^x + e^{2x} = C.$$