Third In-Class Exam Solutions
Math 246, Spring 2008, Professor David Levermore

(1) [10] Consider the matrices

$$\mathbf{A} = \begin{pmatrix} i7 & 2+i \\ 1-i & 4 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}.$$

Compute the matrices

(a) $\mathbf{A}^T$ **Solution.** $\mathbf{A}^T = \begin{pmatrix} i7 & 1-i \\ 2+i & 4 \end{pmatrix}$

(b) $\overline{\mathbf{A}}$ **Solution.** $\overline{\mathbf{A}} = \begin{pmatrix} -i7 & 2-i \\ 1+i & 4 \end{pmatrix}$

(c) $2\mathbf{A}-\mathbf{B}$ **Solution.** $2\mathbf{A}-\mathbf{B} = \begin{pmatrix} i14 & 4+i2 \\ 2-i2 & 8 \end{pmatrix} - \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} -5+i14 & 1+i2 \\ -1-i2 & 6 \end{pmatrix}$

(d) $\mathbf{AB}$ **Solution.** $\mathbf{AB} = \begin{pmatrix} i7 & 2+i \\ 1-i & 4 \end{pmatrix} \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 6+i38 & 4+i23 \\ 17-i5 & 11-i4 \end{pmatrix}$

(e) $\mathbf{B}^{-1}$ **Solution.** Because $\det(\mathbf{B}) = 5\cdot 2 - 3\cdot 3 = 10 - 9 = 1$,

$$\mathbf{B}^{-1} = \frac{1}{\det(\mathbf{B})} \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}.$$

(2) [8] Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}.$$

(a) Find all the eigenvalues of $\mathbf{A}$.

Solution. The characteristic polynomial of $\mathbf{A}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 5z + 4 = (z-1)(z-4).$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are 1 and 4.

(b) For each eigenvalue of $\mathbf{A}$ find an associated eigenvector.

Solution (using the Cayley-Hamilton method from notes). One has

$$\mathbf{A} - \mathbf{I} = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}, \qquad \mathbf{A} - 4\mathbf{I} = \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}.$$

Every nonzero column of $\mathbf{A} - 4\mathbf{I}$ has the form

$$\alpha_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{for some } \alpha_1 \neq 0,$$

any of which is an eigenvector associated with 1. Similarly, every nonzero column of $\mathbf{A} - \mathbf{I}$ has the form

$$\alpha_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{for some } \alpha_2 \neq 0,$$

any of which is an eigenvector associated with 4.

(3) [6] Transform the equation $\dfrac{\mathrm{d}^4u}{\mathrm{d}t^4} + e^t\dfrac{\mathrm{d}^2u}{\mathrm{d}t^2} - 5u = \cos(t)$ into a first-order system of ordinary differential equations.

Solution: Because the equation is fourth order, the first order system must have dimension four. The simplest such first order system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ \cos(t) + 5x_1 - e^t x_3 \end{pmatrix}, \qquad \text{where} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} u \\ u' \\ u'' \\ u''' \end{pmatrix}.$$

(4) [10] Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} t^2+1 \\ t \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t \\ 1 \end{pmatrix}$.

(a) Compute the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2](t)$.

Solution.

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \det\begin{pmatrix} t^2+1 & t \\ t & 1 \end{pmatrix} = t^2 + 1 - t^2 = 1\,.$$

(b) Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to the system $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x}$ wherever $W[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$.

Solution. Let $\mathbf{\Psi}(t) = \begin{pmatrix} t^2+1 & t \\ t & 1 \end{pmatrix}$. Because $\dfrac{\mathbf{\Psi}(t)}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{\Psi}(t)$, one has

$$\begin{aligned} \mathbf{A}(t) &= \frac{\mathbf{\Psi}(t)}{\mathrm{d}t}\mathbf{\Psi}(t)^{-1} = \begin{pmatrix} 2t & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} t^2+1 & t \\ t & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 2t & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -t \\ -t & t^2+1 \end{pmatrix} = \begin{pmatrix} t & 1-t^2 \\ 1 & -t \end{pmatrix}. \end{aligned}$$

(c) Give a general solution to the system you found in part (b).

Solution. Because $W[\mathbf{x}_1, \mathbf{x}_2](t) = 1 \neq 0$, a general solution is

$$\mathbf{x} = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1\begin{pmatrix} t^2+1 \\ t \end{pmatrix} + c_2\begin{pmatrix} t \\ 1 \end{pmatrix}.$$

(5) [10] Compute $e^{t\mathbf{A}}$ for $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ -4 & 1 \end{pmatrix}$.

Solution. The characteristic polynomial of $\mathbf{A}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z - 3 = (z-1)^2 - 4\,.$$

The eigenvalues of $\mathbf{A}$ are therefore 1 ± 2, whereby

$$\begin{aligned} e^{t\mathbf{A}} &= e^t \left[\mathbf{I}\cosh(2t) + (\mathbf{A} - \mathbf{I})\frac{\sinh(2t)}{2} \right] \\ &= e^t \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cosh(2t) + \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix} \frac{\sinh(2t)}{2} \right] \\ &= e^t \begin{pmatrix} \cosh(2t) & -\frac{1}{2}\sinh(2t) \\ -2\sinh(2t) & \cosh(2t) \end{pmatrix}\,. \end{aligned}$$

Alternative Solution. The characteristic polynomial of $\mathbf{A}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z - 3 = (z+1)(z-3)\,.$$

The eigenvalues of $\mathbf{A}$ are therefore -1 and 3. Because

$$\mathbf{A} + \mathbf{I} = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}\,, \qquad \mathbf{A} - 3\mathbf{I} = \begin{pmatrix} -2 & -1 \\ -4 & -2 \end{pmatrix}\,,$$

Eigenpairs of $\mathbf{A}$ are therefore

$$\left(-1\,, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right)\,, \qquad \left(3\,, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right)\,.$$

Set $\mathbf{D} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$ and $\mathbf{V} = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}$. Then $\mathbf{V} = \frac{1}{4}\begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix}$ and

$$\begin{aligned} e^{t\mathbf{A}} &= \mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 2e^{-t} & e^{-t} \\ -2e^{3t} & e^{3t} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2e^{-t} + 2e^{3t} & e^{-t} - e^{3t} \\ 4e^{-t} - 4e^{3t} & 2e^{-t} + 2e^{3t} \end{pmatrix}\,. \end{aligned}$$

(6) [5] Given that $e^{t\mathbf{A}} = \begin{pmatrix} \cos(5t) + \frac{3}{5}\sin(5t) & \frac{4}{5}\sin(5t) \\ \frac{4}{5}\sin(5t) & \cos(5t) - \frac{3}{5}\sin(5t) \end{pmatrix}$, solve the initial-value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix}\,, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}\,.$$

Solution. The solution is

$$\begin{aligned} \mathbf{x}(t) = e^{t\mathbf{A}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} &= \begin{pmatrix} \cos(5t) + \frac{3}{5}\sin(5t) & \frac{4}{5}\sin(5t) \\ \frac{4}{5}\sin(5t) & \cos(5t) - \frac{3}{5}\sin(5t) \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} \cos(5t) - \frac{1}{5}\sin(5t) \\ -\cos(5t) + \frac{7}{5}\sin(5t) \end{pmatrix}\,. \end{aligned}$$

(7) [8] Consider two interconnected tanks filled with brine (salt water). The first tank contains 60 liters and the second contains 40 liters. Brine flows with a concentration of 3 grams of salt per liter flows into the first tank at a rate of 5 liters per hour. Well stirred brine flows from the first tank to the second at a rate of 7 liters per hour, from the second to the first at a rate of 2 liters per hour, and from the second into a drain at a rate of 5 liters per hour. At $t = 0$ there are 40 grams of salt in the first tank and 20 grams in the second. Give an initial-value problem that governs the amount of salt in each tank as a function of time.

Solution. The rates work out so there will always be 60 liters of brine in the first tank and 40 liters in the second. Let $S_1(t)$ and $S_2(t)$ be the grams of salt in the first and second tanks respectively. These are governed by the initial-value problem

$$\begin{aligned} \frac{\mathrm{d}S_1}{\mathrm{d}t} &= 3\cdot 5 + \frac{S_2}{40}\,2 - \frac{S_1}{60}\,7\,, \qquad && S_1(0) = 40\,,\\ \frac{\mathrm{d}S_2}{\mathrm{d}t} &= \frac{S_1}{60}\,7 - \frac{S_2}{40}\,2 - \frac{S_2}{40}\,5\,, \qquad && S_2(0) = 20\,. \end{aligned}$$

(8) [10] Consider the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y^2 \\ x - 6y + xy \end{pmatrix}.$$

(a) Find all of its stationary points.

Solution. Stationary points satisfy

$$0 = x - y^2\,, \qquad 0 = x - 6y + xy\,.$$

When the first equation is used to eliminate x in the second equation, one finds

$$0 = y^3 + y^2 - 6y = y(y-2)(y+3)\,,$$

whereby either $y = 0$, $y = 2$, or $y = -3$. All the stationary points are therefore

$$(0,0)\,, \qquad (4,2)\,, \qquad (9,-3)\,.$$

(b) Compute the coefficient matrix of the linearization associated with each stationary point.

Solution. Because

$$\begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = \begin{pmatrix} x - y^2 \\ x - 6y + xy \end{pmatrix},$$

the matrix of partial derivatives is

$$\begin{pmatrix} \partial_x f(x,y) & \partial_y f(x,y) \\ \partial_x g(x,y) & \partial_y g(x,y) \end{pmatrix} = \begin{pmatrix} 1 & -2y \\ 1+y & -6+x \end{pmatrix}.$$

Evaluating this matrix at each stationary point yields the coefficient matrices

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} 1 & 0 \\ 1 & -6 \end{pmatrix} \qquad && \text{at } (0,0)\,,\\ \mathbf{A} &= \begin{pmatrix} 1 & -4 \\ 3 & -2 \end{pmatrix} \qquad && \text{at } (4,2)\,,\\ \mathbf{A} &= \begin{pmatrix} 1 & 6 \\ -2 & 3 \end{pmatrix} \qquad && \text{at } (9,-3)\,. \end{aligned}$$

(9) [16] Find a general solution for each of the following systems.

(a) $$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 9\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $\pm i3$. One therefore has

$$\begin{aligned} e^{t\mathbf{A}} &= \left[\mathbf{I}\cos(3t) + \mathbf{A}\frac{\sin(3t)}{3}\right] = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(3t) + \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix}\frac{\sin(3t)}{3}\right] \\ &= \begin{pmatrix} \cos(3t) + \frac{1}{3}\sin(3t) & \frac{2}{3}\sin(3t) \\ -\frac{5}{3}\sin(3t) & \cos(3t) - \frac{1}{3}\sin(3t) \end{pmatrix}. \end{aligned}$$

A general solution is therefore given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1\begin{pmatrix} \cos(3t) + \frac{1}{3}\sin(3t) \\ -\frac{5}{3}\sin(3t) \end{pmatrix} + c_2\begin{pmatrix} \frac{2}{3}\sin(3t) \\ \cos(3t) - \frac{1}{3}\sin(3t) \end{pmatrix}.$$

(b) $$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 0 & 2 \\ -4 & -4 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 4z + 8 = (z+2)^2 + 2^2\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $-2 \pm i2$. One therefore has

$$\begin{aligned} e^{t\mathbf{A}} &= e^{-2t}\left[\mathbf{I}\cos(2t) + (\mathbf{A} + 2\mathbf{I})\frac{\sin(2t)}{2}\right] \\ &= e^{-2t}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(2t) + \begin{pmatrix} 2 & 2 \\ -4 & -2 \end{pmatrix}\frac{\sin(2t)}{2}\right] \\ &= e^{-2t}\begin{pmatrix} \cos(2t) + \sin(2t) & \sin(2t) \\ -2\sin(2t) & \cos(2t) - \sin(2t) \end{pmatrix}. \end{aligned}$$

A general solution is therefore given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-2t}\begin{pmatrix} \cos(2t) + \sin(2t) \\ -2\sin(2t) \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} \sin(2t) \\ \cos(2t) - \sin(2t) \end{pmatrix}.$$

(10) [8] Sketch the phase portrait for each of the systems in the previous problem. Identify the type and stability of the origin.

Solution. The characteristic polynomial of $\mathbf{A}$ is $p(z) = z^2 + 9$. Because $\mu = 0$, $\delta = -9 < 0$, and $a_{21} < 0$ the phase portrait is a *clockwise center*. The origin is thereby *stable*. The phase portrait should indicate a family of clockwise elliptical trajectories that go around the origin.

Solution. The characteristic polynomial of $\mathbf{A}$ is $p(z) = (z+2)^2 + 4$. Because $\mu = -2$, $\delta = -4 < 0$, and $a_{21} < 0$ the phase portrait is a *clockwise spiral sink*. The origin is thereby *asymptotically stable*. The phase portrait should indicate a family of clockwise spiral trajectories that approach the origin.

(11) [9] Suppose you know that for some first-order planar system of nonlinear ordinary differential equations:

- its stationary points are $(0,0)$, $(2,2)$, and $(4,0)$;
- for $(0,0)$ the coefficient matrix of the linearization has eigenvalues 1 and 2 with respective eigenvectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} ;$$

- for $(2,2)$ the coefficient matrix of the linearization has eigenvalues -1 and -2 with respective eigenvectors

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} ;$$

- for $(4,0)$ the coefficient matrix of the linearization has eigenvalues 1 and -1 with respective eigenvectors

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} .$$

Sketch a plausible phase portrait for the system. Identify the type and stability of each stationary point.

Solution. For each stationary point you have the following.

- The stationary point $(0,0)$ has two positive simple real eigenvalues. It therefore is a *nodal source* and thereby is *unstable*. Near it there is one trajectory that emerges from $(0,0)$ tangent to each side of the line $y = x$. Every other trajectory emerges from $(0,0)$ tangent to the line $y = 0$.
- The stationary point $(2,2)$ has two negative simple real eigenvalues. It therefore is a *nodal sink* and thereby is *asymptotically stable*. Near it there is one trajectory that approaches $(2,2)$ tangent to each side of the line $y = x$. Every other trajectory approaches $(2,2)$ tangent to the line $y = -x + 4$.
- The stationary point $(4,0)$ has one negative and one positive real eigenvalue. It therefore is a *saddle* and thereby is *unstable*. Near it there is one trajectory that emerges from $(4,0)$ tangent to each side of the line $y = -x + 4$. There is also one trajectory that approaches $(4,0)$ tangent to each side of the line $y = 0$.