Solutions of Sample Problems for Second In-Class Exam
Math 246, Spring 2008, Professor David Levermore

(1) Give the interval of existence for the solution of the initial-value problem

$$\frac{\mathrm{d}^3x}{\mathrm{d}t^3} + \frac{\cos(3t)}{4-t}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{e^{-2t}}{1+t}, \qquad x(2) = x'(2) = x''(2) = 0\,.$$

Solution. The coefficient and forcing are both continuous over the interval $(-1, 4)$, which contains the initial time $t = 2$. The coefficient is not defined at $t = 4$ while the forcing is not defined at $t = -1$. The interval of existence is therefore $(-1, 4)$.

(2) Let $\mathbf{L}$ be a linear ordinary differential operator with constant coefficients. Suppose that all the roots of its characteristic polynomial (listed with their multiplicities) are $-2 + i3$, $-2 - i3$, $i7$, $i7$, $-i7$, $-i7$, 5, 5, 5, -3, 0, 0.

(a) Give the order of $\mathbf{L}$.

Solution. There are 12 roots listed, so the degree of the characteristic polynomial is 12, whereby the order of $\mathbf{L}$ is 12.

(b) Give a general real solution of the homogeneous equation $\mathbf{L}y = 0$.

Solution. A general solution is

$$\begin{aligned} y = {} & c_1 e^{-2t}\cos(3t) + c_2 e^{-2t}\sin(3t) \\ & + c_3\cos(7t) + c_4\sin(7t) + c_5 t\,\cos(7t) + c_6 t\,\sin(7t) \\ & + c_7 e^{5t} + c_8 t\,e^{5t} + c_9 t^2 e^{5t} + c_{10} e^{-3t} + c_{11} + c_{12} t\,. \end{aligned}$$

The reasoning is as follows:

- the single conjugate pair $-2 \pm i3$ yields $e^{-2t}\cos(3t)$ and $e^{-2t}\sin(3t)$;
- the double conjugate pair $-2 \pm i3$ yields

$$\cos(7t)\,, \quad \sin(7t)\,, \quad t\,\cos(7t)\,, \quad \text{and} \quad t\,\sin(7t)\,;$$

- the triple real root 5 yields e^{5t}, $t\,e^{5t}$, and t^2e^{5t};
- the single real root -3 yields e^{-3t};
- the double real root 0 yields 1 and t.

(3) Let $\mathbf{D} = \dfrac{\mathrm{d}}{\mathrm{d}t}$. Solve each of the following initial-value problems.

(a) $\mathbf{D}^2y + 4\mathbf{D}y + 4y = 0\,, \qquad y(0) = 1\,, \quad y'(0) = 0\,.$

Solution. This is a constant coefficient, homogeneous, linear equation. Its characteristic polynomial is

$$P(z) = z^2 + 4z + 4 = (z+2)^2\,.$$

This has the double real root -2, which yields a general solution

$$y(t) = c_1 e^{-2t} + c_2 t\,e^{-2t}\,.$$

Because

$$y'(t) = -2c_1 e^{-2t} - 2c_2 t\,e^{-2t} + c_2 e^{-2t}\,,$$

when the initial conditions are imposed, one finds that

$$y(0) = c_1 = 1\,, \qquad y'(0) = -2c_1 + c_2 = 0\,.$$

These are solved to find $c_1 = 1$ and $c_2 = 2$. The solution of the initial-value problem is therefore

$$y(t) = e^{-2t} + 2t\,e^{-2t} = (1 + 2t)e^{-2t}\,.$$

(b) $\mathbf{D}^2 y + 9y = 20e^t\,, \qquad y(0) = 0\,, \quad y'(0) = 0\,.$

Solution. This is a constant coefficient, inhomogeneous, linear equation. Its characteristic polynomial is

$$P(z) = z^2 + 9 = z^2 + 3^2\,.$$

This has the conjugate pair of roots $\pm i3$, which yields a general solution of the associated homogeneous problem

$$y_H(t) = c_1 \cos(3t) + c_2 \sin(3t)\,.$$

The forcing $20e^t$ has degree $d = 0$ and characteristic $r + is = 1$, which is a root of $P(z)$ of multiplicity $m = 0$. A particular solution $y_P(t)$ can be found by the method of undetermined coefficients using either direct substitution (as in the book) or KEY identity evaluation (as in the lectures).

Direct Substitution. Because $m = d = 0$, you seek a particular solution of the form

$$y_P(t) = Ae^t\,.$$

Because

$$y_P'(t) = Ae^t\,, \qquad y_P''(t) = Ae^t\,,$$

one sees that

$$\mathbf{L}y_P(t) = y_P''(t) + 9y_P(t) = Ae^t + 9Ae^t = 10Ae^t\,.$$

Setting $\mathbf{L}y_P(t) = 10Ae^t = 20e^t$, we see that $A = 2$. Hence, $y_P(t) = 2e^t$.

KEY Indentity Evaluations. Because $m + d = 0$, you only need to evaluate the KEY identity at $z = 1$, to find

$$\mathbf{L}(e^t) = P(1)e^t = (1^2 + 9)e^t = 10e^t\,.$$

Multiplying this equation by 2 yields $\mathbf{L}(2e^t) = 20e^t$. Hence, $y_P(t) = 2e^t$.

By either approach one finds $y_P(t) = 2e^t$, which yields the general solution

$$y(t) = c_1 \cos(3t) + c_2 \sin(3t) + 2e^t\,.$$

Because

$$y'(t) = -3c_1 \sin(3t) + 3c_2 \cos(3t) + 2e^t\,,$$

when the initial conditions are imposed, one finds that

$$y(0) = c_1 + 2 = 0\,, \qquad y'(0) = 3c_2 + 2 = 0\,.$$

These are solved to find $c_1 = -2$ and $c_2 = -\frac{2}{3}$. The solution of the initial-value problem is therefore

$$y(t) = -2\cos(3t) - \tfrac{2}{3}\sin(3t) + 2e^t\,.$$

(4) Let $\mathbf{D} = \dfrac{\mathrm{d}}{\mathrm{d}t}$. Give a general real solution for each of the following equations.

(a) $\mathbf{D}^2 y + 4\mathbf{D}y + 5y = 3\cos(2t)$.

Solution. This is a constant coefficient, inhomogeneous, linear equation. Its characteristic polynomial is

$$P(z) = z^2 + 4z + 5 = (z+2)^2 + 1\,.$$

This has the conjugate pair of roots $-2 \pm i$, which yields a general solution of the associated homogeneous problem

$$y_H(t) = c_1 e^{-2t}\cos(t) + c_2 e^{-2t}\sin(t)\,.$$

The forcing $3\cos(2t)$ has degree $d = 0$ and characteristic $r + is = i2$, which is a root of $P(z)$ of multiplicity $m = 0$. A particular solution $y_P(t)$ can be found by the method of undetermined coefficients using either direct substitution (as in the book) or KEY identity evaluation (as in the lectures).

Direct Substitution. Because $m = d = 0$, you seek a particular solution of the form

$$y_P(t) = A\cos(2t) + B\sin(2t)\,.$$

Because

$$\begin{aligned} y_P'(t) &= -2A\sin(2t) + 2B\cos(2t)\,,\\ y_P''(t) &= -4A\cos(2t) - 4B\sin(2t)\,, \end{aligned}$$

one sees that

$$\begin{aligned} \mathbf{L}y_P(t) &= y_P''(t) + 4y_P'(t) + 5y_P(t)\\ &= \big[-4A\cos(2t) - 4B\sin(2t)\big] + 4\big[-2A\sin(2t) + 2B\cos(2t)\big]\\ &\quad + 5\big[A\cos(2t) + B\sin(2t)\big]\\ &= (A + 8B)\cos(2t) + (B - 8A)\sin(2t)\,. \end{aligned}$$

Setting $\mathbf{L}y_P(t) = (A + 8B)\cos(2t) + (B - 8A)\sin(2t) = 3\cos(2t)$, we see that

$$A + 8B = 3\,, \qquad B - 8A = 0\,.$$

One finds that $A = \frac{3}{65}$ and $B = \frac{24}{65}$. Hence, $y_P(t) = \frac{3}{65}\cos(2t) + \frac{24}{65}\sin(2t)$. A general solution is therefore

$$y = c_1 e^{-2t}\cos(t) + c_2 e^{-2t}\sin(t) + \tfrac{3}{65}\cos(2t) + \tfrac{24}{65}\sin(2t)\,.$$

KEY Indentity Evaluations. Because $m + d = 0$, you only need to evaluate the KEY identity at $z = i2$, to find

$$\mathbf{L}\big(e^{i2t}\big) = P(i2)e^{i2t} = \big((i2)^2 + 4(i2) + 5\big)e^{i2t} = (1 + i8)e^{i2t}\,.$$

Because the forcing $3\cos(2t) = 3\,\mathrm{Re}\big(e^{i2t}\big)$, you divide the above by $1 + i8$ and multiply by 3 to find

$$\mathbf{L}\left(\frac{3}{1+i8}\,e^{i2t}\right) = 3e^{i2t}\,.$$

Hence,

$$y_P(t) = \mathrm{Re}\bigg(\frac{3}{1+i8}\,e^{i2t}\bigg) = \mathrm{Re}\bigg(\frac{3(1-i8)}{1^2+8^2}\,e^{i2t}\bigg) = \tfrac{3}{65}\,\mathrm{Re}\big((1-i8)e^{i2t}\big)$$
$$= \tfrac{3}{65}\big(\cos(2t) + 8\sin(2t)\big) = \tfrac{3}{65}\cos(2t) + \tfrac{24}{65}\sin(2t)\,.$$

A general solution is therefore

$$y = c_1 e^{-2t}\cos(t) + c_2 e^{-2t}\sin(t) + \tfrac{3}{65}\cos(2t) + \tfrac{24}{65}\sin(2t)\,.$$

(b) $\mathbf{D}^2 y - y = e^t$.

Solution. This is a constant coefficient, inhomogeneous, linear equation. Its characteristic polynomial is

$$P(z) = z^2 - 1 = (z+1)(z-1)\,.$$

This has the real roots -1 and 1, which yields a general solution of the associated homogeneous problem

$$y_H(t) = c_1 e^{-t} + c_2 e^t\,.$$

The forcing e^t has degree $d = 0$ and characteristic $r + is = 1$, which is a root of $P(z)$ of multiplicity $m = 1$. A particular solution $y_P(t)$ can be found by the method of undetermined coefficients using either direct substitution (as in the book) or KEY identity evaluation (as in the lectures).

Direct Substitution. Because $m = 1$ and $d = 0$, you seek a particular solution of the form

$$y_P(t) = At\,e^t\,,$$

Because

$$y_P'(t) = At\,e^t + Ae^t\,,$$
$$y_P''(t) = At\,e^t + 2Ae^t\,,$$

one sees that

$$\mathbf{L}y_P(t) = y_P''(t) - y_P(t) = \big[At\,e^t + 2Ae^t\big] - \big[At\,e^t\big] = 2Ae^t\,.$$

Setting $\mathbf{L}y_P(t) = 2Ae^t = e^t$, we see that $A = \frac{1}{2}$. Hence, $y_P(t) = \frac{1}{2}t\,e^t$. A general solution is therefore

$$y = c_1 e^{-t} + c_2 e^t + \tfrac{1}{2}t\,e^t\,.$$

KEY Indentity Evaluations. Because $m + d = 1$, you need the KEY identity and its first derivative

$$\mathbf{L}\big(e^{zt}\big) = (z^2 - 1)e^{zt}\,, \qquad \mathbf{L}\big(t\,e^{zt}\big) = (z^2-1)t\,e^{zt} + 2ze^{zt}\,.$$

Evaluate these at $z = 1$ to find

$$\mathbf{L}\big(e^t\big) = 0\,, \qquad \mathbf{L}\big(t\,e^t\big) = 2e^t\,.$$

Dividing the last equation by 2 yields $\mathbf{L}(\frac{1}{2}t\,e^t) = e^t$. Hence, $y_P(t) = \frac{1}{2}t\,e^t$. A general solution is therefore

$$y = c_1 e^{-t} + c_2 e^t + \tfrac{1}{2}t\,e^t\,.$$

(5) The functions x and x^2 are solutions of the homogeneous equation

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0 \qquad \text{over } x > 0\,.$$

(You do not have to check that this is true!)

(a) Compute their Wronskian.

Solution. The Wronskian is

$$W[x, x^2](x) = \det\begin{pmatrix} x & x^2 \\ 1 & 2x \end{pmatrix} = x(2x) - 1x^2 = 2x^2 - x^2 = x^2\,.$$

(b) Give a general solution of the equation

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = x^3e^x \qquad \text{over } x > 0\,.$$

You may express the solution in terms of definite integrals.

Solution. Because $W[x, x^2](x) = x^2 \neq 0$ over $x > 0$, the functions x and x^2 are linearly independent. A general solution of the associated homogeneous problem is

$$y_H(x) = c_1x + c_2x^2\,.$$

Because this problem does not have constant coefficients, you must use the method of variation of parameters to find a particular solution $y_P(x)$. First, divide by x^2 to bring the equation into its normal form

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{2}{x}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x^2}\,y = xe^x \qquad \text{over } x > 0\,.$$

Seek a solution in the form

$$y = u_1(x)x + u_2(x)x^2\,.$$

where $u_1'(x)$ and $u_2'(x)$ satisfy

$$\begin{aligned} u_1'(x)x + u_2'(x)x^2 &= 0\,, \\ u_1'(x)1 + u_2'(x)2x &= x\,e^x\,. \end{aligned}$$

Solve this system to find

$$u_1'(x) = -x\,e^x\,, \qquad u_2'(x) = e^x\,.$$

Integrate these equations to find

$$u_1(x) = c_1 + (1-x)e^x\,, \qquad u_2(x) = c_2 + e^x\,.$$

A general solutions is therefore

$$y = c_1x + c_2x^2 + (1-x)e^xx + e^xx^2 = c_1x + c_2x^2 + x\,e^x\,.$$

(6) What answer will be produced by the following MATLAB commands?

```
>> ode1 = 'D2y + 2*Dy + 5*y = 16*exp(t)';
>> dsolve(ode1, 't')
ans =
```

Solution. The commands ask MATLAB to give the general solution of the equation

$$\mathbf{D}^2 y + 2\mathbf{D}y + 5y = 16e^t\,, \qquad \text{where} \quad \mathbf{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

MATLAB will produce the answer

2*exp(t) + C1*exp(–t)*sin(2*t) + C2*exp(–t)*cos(2*t)

This can be seen as follows. This is a constant coefficient, inhomogeneous, linear equation. The characteristic polynomial is

$$P(z) = z^2 + 2z + 5 = (z+1)^2 + 4 = (z+1)^2 + 2^2\,.$$

Its roots are the conjugate pair $-1 \pm i2$. A general solution of the associated homogeneous problem is

$$y_H(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)\,.$$

The forcing $16e^t$ has degree $d = 0$ and characteristic $r + is = 1$, which is a root of $P(z)$ of multiplicity $m = 0$. A particular solution $y_P(t)$ can be found by the method of undetermined coefficients using either direct substitution (as in the book) or KEY identity evaluation (as in the lectures).

Direct Substitution. Because $m = d = 0$, you seek a particular solution of the form

$$y_P(t) = Ae^t\,.$$

Because

$$y_P'(t) = Ae^t\,, \qquad y_P''(t) = Ae^t\,,$$

one sees that

$$\mathbf{L}y_P(t) = y_P''(t) + 2y_P'(t) + 5y_P(t) = [Ae^t] + 2[Ae^t] + 5[Ae^t] = 8Ae^t\,.$$

Setting $\mathbf{L}y_P(t) = 8Ae^t = 16e^t$, we see that $A = 2$. Hence, $y_P(t) = 2e^t$.

KEY Indentity Evaluations. Because $m + d = 0$, you only need to evaluate the KEY identity at $z = 1$, to find

$$\mathbf{L}(e^t) = P(1)e^t = (1^2 + 2\cdot 1 + 5)e^t = 8e^t\,.$$

Multiply this by 2 to obtain $\mathbf{L}(2e^t) = 16e^t$. Hence, $y_P(t) = 2e^t$.

By either approach you find $y_P(t) = 2e^t$. A general solution is therefore

$$y = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t) + 2e^t\,.$$

Up to notational differnces, this is the answer that MATLAB produces.

(7) The vertical displacement of a mass on a spring is given by

$$z(t) = \sqrt{3}\cos(2t) + \sin(2t)\,.$$

Express this in the form $z(t) = A\cos(\omega t - \delta)$, identifying the amplitude and phase of the oscillation.

Solution. The displacement takes the form

$$z(t) = 2\cos(2t - \tfrac{\pi}{6})\,,$$

where the amplitude is 2 and the phase is $\frac{\pi}{6}$. There are several approaches to the problem. Here are two.

One approach that requires no memorization other than the addition formula for cosine is the following. Because

$$A\cos(\omega t - \delta) = A\cos(\delta)\cos(\omega t) + A\sin(\delta)\sin(\omega t)\,,$$

this form will be equal to $z(t)$ provided $\omega = 2$ and

$$A\cos(\delta) = \sqrt{3}\,, \qquad A\sin(\delta) = 1\,.$$

Upon solving these equations, one finds that the amplitude A is given by

$$A = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2\,,$$

while the phase δ is in the first quadrant and is given by (for example)

$$\delta = \sin^{-1}\left(\frac{1}{A}\right) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}\,.$$

Another approach requires you to memorize special formulas for both the amplitude and phase of functions of the form

$$c_1\cos(\omega t) + c_2\sin(\omega t)\,.$$

The formula for the amplitude is easy to remember because c_1 and c_2 appear in it symmetrically. It gives

$$A = \sqrt{c_1^2 + c_2^2} = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2\,.$$

Formulas for phase are trickier because c_1 and c_2 do not play symmetric roles. Because both c_1 and c_2 are positive, δ is in the first quadrant and is given by

$$\delta = \cos^{-1}\left(\frac{c_2}{A}\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}\,.$$

This formula holds whenever $c_1 \geq 0$, otherwise its result must be subtracted from 2π. The most common mistake made by those who choose this approach is to confuse the roles of c_1 and c_2. One way to keep these roles straight is to remember this formula verbally as

$$\text{phase} = \begin{cases} \cos^{-1}\left(\dfrac{\text{coefficient of cosine}}{\text{amplitude}}\right) & \text{for coefficient of sine nonnegative}\,, \\ 2\pi - \cos^{-1}\left(\dfrac{\text{coefficient of cosine}}{\text{amplitude}}\right) & \text{for coefficient of sine negative}\,. \end{cases}$$

(8) When a mass of 4 grams is hung vertically from a spring, at rest it stretches the spring 9.8 cm. (Gravitational acceleration is $g = 980$ cm/sec^2.) At $t = 0$ the mass is displaced 3 cm above its equilibrium position and is released with no initial velocity. It moves in a medium that imparts a drag force of 2 dynes (1 dyne = 1 gram cm/sec^2) when the speed of the mass is 4 cm/sec. There are no other forces. (Assume that the spring force is proportional to displacement and that the drag force is proportional to velocity.)

(a) Formulate an initial-value problem that governs the motion of the mass for $t > 0$. (DO NOT solve this initial-value problem, just write it down!)

Solution. Let $h(t)$ be the displacement of the mass from its equilibrium (rest) position at time t in centimeters, with upward displacements being positive. The governing initial-value problem then has the form

$$m\frac{\mathrm{d}^2h}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}h}{\mathrm{d}t} + kh = 0\,, \qquad h(0) = 3\,, \quad h'(0) = 0\,,$$

where m is the mass, γ is the drag coefficient, and k is the spring constant. The problem says that $m = 4$ grams. The spring constant is obtained by balancing the weight of the mass ($mg = 4 \cdot 980$ dynes) with the force applied by the spring when it is stretched 9.8 cm. This gives $k\,9.8 = 4 \cdot 980$, or

$$k = \frac{4 \cdot 980}{9.8} = 400 \quad \text{dynes/cm}\,.$$

The drag coefficient is obtained by balanceing the force of 2 dynes with the drag force imparted by the medium when the speed of the mass is 4 cm/sec. This gives $\gamma\,4 = 2$, or

$$\gamma = \frac{2}{4} = \frac{1}{2} \quad \text{dynes sec/cm}\,.$$

The governing initial-value problem is therefore

$$4\frac{\mathrm{d}^2h}{\mathrm{d}t^2} + \frac{1}{2}\frac{\mathrm{d}h}{\mathrm{d}t} + 400h = 0\,, \qquad h(0) = 3\,, \quad h'(0) = 0\,,$$

If you had chosen downward displacements to be positive then the governing initial-value problem would be the same except for the first initial condition, which would then be $h(0) = -3$.

(b) What is the natural frequency of the spring?

Solution. The natural frequency of the spring is given by

$$\omega_o = \sqrt{\frac{k}{m}} = \sqrt{\frac{4 \cdot 980}{4 \cdot 9.8}} = \sqrt{100} = 10 \quad 1/\text{sec}\,.$$

(c) Show that the system is under damped and find its quasifrequency.

Solution. The characteristic polynomial is

$$P(z) = z^2 + \frac{1}{8}z + 100 = \left(z + \frac{1}{16}\right)^2 + 100 - \frac{1}{16^2}\,,$$

which has a conjugate pair of roots. The system is therefore under damped. The roots are $-\frac{1}{16} \pm i\mu$ where

$$\mu = \sqrt{100 - \frac{1}{16^2}}\,.$$

This is the quasifrequency.

(9) Compute the Laplace transform of $f(t) = t\,e^{3t}$ from its definition.

Solution. The definition of Laplace transform gives

$$\mathcal{L}[f](s) = \lim_{T\to\infty} \int_0^T e^{-st} t\, e^{3t}\, dt = \lim_{T\to\infty} \int_0^T t\, e^{(3-s)t}\, dt\,.$$

This limit diverges to $+\infty$ for $s \leq 3$ because in that case

$$\int_0^T t\, e^{(3-s)t}\, dt \geq \int_0^T t\, dt = \frac{T^2}{2}\,,$$

which clearly diverges to $+\infty$ as $T \to \infty$.

For $s > 3$ an integration by parts shows that

$$\begin{aligned}
\int_0^T t\, e^{(3-s)t}\, dt &= t\,\frac{e^{(3-s)t}}{3-s}\bigg|_0^T - \int_0^T \frac{e^{(3-s)t}}{3-s}\, dt \\
&= \left(t\,\frac{e^{(3-s)t}}{3-s} - \frac{e^{(3-s)t}}{(3-s)^2}\right)\bigg|_0^T \\
&= \left(T\,\frac{e^{(3-s)T}}{3-s} - \frac{e^{(3-s)T}}{(3-s)^2}\right) + \frac{1}{(3-s)^2}\,.
\end{aligned}$$

Hence, for $s > 3$ one has that

$$\begin{aligned}
\mathcal{L}[f](s) &= \lim_{T\to\infty} \left[\left(T\,\frac{e^{(3-s)T}}{3-s} - \frac{e^{(3-s)T}}{(3-s)^2}\right) + \frac{1}{(3-s)^2}\right] \\
&= \frac{1}{(3-s)^2} + \lim_{T\to\infty} \left(T\,\frac{e^{(3-s)T}}{3-s} - \frac{e^{(3-s)T}}{(3-s)^2}\right) \\
&= \frac{1}{(3-s)^2}\,.
\end{aligned}$$

(10) Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 4\frac{\mathrm{d}y}{\mathrm{d}t} + 13y = f(t)\,, \qquad y(0) = 4\,, \quad y'(0) = 1\,,$$

where

$$f(t) = \begin{cases} \cos(t) & \text{for } 0 \le t < 2\pi\,, \\ t - 2\pi & \text{for } t \ge 2\pi\,. \end{cases}$$

You may refer to the table on the last page. DO NOT take the inverse Laplace transform to find $y(t)$, just solve for $Y(s)$!

Solution. The Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) + 4\mathcal{L}[y'](s) + 13\mathcal{L}[y](s) = \mathcal{L}[f](s)\,,$$

where

$$\begin{aligned} \mathcal{L}[y](s) &= Y(s)\,, \\ \mathcal{L}[y'](s) &= sY(s) - y(0) = sY(s) - 4\,, \\ \mathcal{L}[y''](s) &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 4s - 1\,. \end{aligned}$$

To compute $\mathcal{L}[f](s)$, first write f as

$$\begin{aligned} f(t) &= \big(1 - u(t-2\pi)\big)\cos(t) + u(t-2\pi)(t-2\pi) \\ &= \cos(t) - u(t-2\pi)\cos(t) + u(t-2\pi)(t-2\pi) \\ &= \cos(t) - u(t-2\pi)\cos(t-2\pi) + u(t-2\pi)(t-2\pi)\,. \end{aligned}$$

Referring to the table on the last page, item 5 with $c = 2\pi$, item 2 with $b = 1$, and item 1 with $n = 1$ then show that

$$\begin{aligned} \mathcal{L}[f](s) &= \mathcal{L}[\cos(t)](s) - \mathcal{L}[u(t-2\pi)\cos(t-2\pi)](s) + \mathcal{L}[u(t-2\pi)(t-2\pi)](s) \\ &= \mathcal{L}[\cos(t)](s) - e^{-2\pi s}\mathcal{L}[\cos(t)](s) + e^{-2\pi s}\mathcal{L}[t](s) \\ &= \big(1 - e^{-2\pi s}\big)\frac{s}{s^2+1} + e^{-2\pi s}\frac{1}{s^2}\,. \end{aligned}$$

The Laplace transform of the initial-value problem then becomes

$$\big(s^2Y(s) - 4s - 1\big) + 4\big(sY(s) - 4\big) + 13Y(s) = \big(1 - e^{-2\pi s}\big)\frac{s}{s^2+1} + e^{-2\pi s}\frac{1}{s^2}\,,$$

which becomes

$$(s^2 + 4s + 13)Y(s) - 4s - 1 - 16 = \big(1 - e^{-2\pi s}\big)\frac{s}{s^2+1} + e^{-2\pi s}\frac{1}{s^2}\,.$$

Hence, $Y(s)$ is given by

$$Y(s) = \frac{1}{s^2 + 4s + 13}\left(4s + 17 + \big(1 - e^{-2\pi s}\big)\frac{s}{s^2+1} + e^{-2\pi s}\frac{1}{s^2}\right).$$

(11) Find the inverse Laplace transforms of the following functions. You may refer to the table on the last page.

(a) $F(s) = \dfrac{2}{(s+5)^2}$,

Solution. Referring to the table on the last page, item 1 with $n = 1$ gives $\mathcal{L}[t](s) = 1/s^2$. Item 4 with $a = -5$ and $f(t) = t$ then gives

$$\mathcal{L}[e^{-5t}t](s) = \frac{1}{(s+5)^2}\,.$$

Multiplying this by 2 yields

$$\mathcal{L}[2e^{-5t}t](s) = \frac{2}{(s+5)^2}\,.$$

You therefore conclude that

$$\mathcal{L}^{-1}\left[\frac{2}{(s+5)^2}\right](t) = 2e^{-5t}t\,.$$

(b) $F(s) = \dfrac{3s}{s^2 - s - 6}$,

Solution. The denominator factors as $(s-3)(s+2)$, so the partial fraction decomposition is

$$\frac{3s}{s^2 - s - 6} = \frac{3s}{(s-3)(s+2)} = \frac{\frac{9}{5}}{s-3} + \frac{\frac{6}{5}}{s+2}\,.$$

Referring to the table on the last page, item 1 with $n = 0$ gives $\mathcal{L}[1](s) = 1/s$. Item 4 with $a = 3$ and $f(t) = 1$, and with $a = -2$ and $f(t) = 1$, then gives

$$\mathcal{L}[e^{3t}](s) = \frac{1}{s-3}\,, \qquad \mathcal{L}[e^{-2t}](s) = \frac{1}{s+2}\,,$$

whereby

$$\frac{3s}{s^2 - s - 6} = \tfrac{9}{5}\mathcal{L}[e^{3t}](s) + \tfrac{6}{5}\mathcal{L}[e^{-2t}](s) = \mathcal{L}\left[\tfrac{9}{5}e^{3t} + \tfrac{6}{5}e^{-2t}\right](s)\,.$$

You therefore conclude that

$$\mathcal{L}^{-1}\left[\frac{3s}{s^2 - s - 6}\right](t) = \tfrac{9}{5}e^{3t} + \tfrac{6}{5}e^{-2t}\,.$$

(c) $F(s) = \dfrac{(s-2)e^{-3s}}{s^2 - 4s + 5}$.

Solution. Complete the square in the denominator to get $(s-2)^2+1$. Referring to the table on the last page, item 2 with $b = 1$ gives

$$\mathcal{L}[\cos(t)](s) = \frac{s}{s^2+1}\,.$$

Item 4 with $a = 2$ and $f(t) = \cos(t)$ then gives

$$\mathcal{L}[e^{2t}\cos(t)](s) = \frac{s-2}{(s-2)^2+1}\,.$$

Item 5 with $c = 3$ and $f(t) = e^{2t}\cos(t)$ then gives

$$\mathcal{L}[u(t-3)e^{2(t-3)}\cos(t-3)](s) = e^{-3s}\frac{s-2}{(s-2)^2+1}\,.$$

You therefore conclude that

$$\mathcal{L}^{-1}\left[e^{-3s}\frac{s-2}{s^2-4s+5}\right](t) = u(t-3)e^{2(t-3)}\cos(t-3)\,.$$

A Short Table of Laplace Transforms

$\mathcal{L}[t^n](s) = \dfrac{n!}{s^{n+1}}$	for $s > 0$.
$\mathcal{L}[\cos(bt)](s) = \dfrac{s}{s^2+b^2}$	for $s > 0$.
$\mathcal{L}[\sin(bt)](s) = \dfrac{b}{s^2+b^2}$	for $s > 0$.
$\mathcal{L}[e^{at}f(t)](s) = F(s-a)$	where $F(s) = \mathcal{L}[f(t)](s)$.
$\mathcal{L}[u(t-c)f(t-c)](s) = e^{-cs}F(s)$	where $F(s) = \mathcal{L}[f(t)](s)$ and u is the step function.