Sample Problems for Third In-Class Exam
Math 246, Spring 2008, Professor David Levermore

(1) Consider the matrices

$$\mathbf{A} = \begin{pmatrix} -i2 & 1+i \\ 2+i & -4 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix}.$$

Compute the matrices
(a) $\mathbf{A}^T$,
(b) $\overline{\mathbf{A}}$,
(c) $\mathbf{A}^*$,
(d) $5\mathbf{A} - \mathbf{B}$,
(e) $\mathbf{AB}$,
(f) $\mathbf{B}^{-1}$.

(2) Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 3 \\ 4 & -1 \end{pmatrix}.$$

(a) Find all the eigenvalues of $\mathbf{A}$.
(b) For each eigenvalue of $\mathbf{A}$ find all of its eigenvectors.

(3) Solve each of the following initial-value problems.

(a) $$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(b) $$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(4) Compute $e^{t\mathbf{A}}$ for $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$.

(5) Find a general solution for each of the following systems.

(a) $$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

(b) $$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

(c) $$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

(6) Sketch the phase portrait for each of the systems in Problem 5.

(7) Transform the equation $\dfrac{\mathrm{d}^3u}{\mathrm{d}t^3} + t^2\dfrac{\mathrm{d}u}{\mathrm{d}t} - 3u = \sinh(2t)$ into a first-order system of ordinary differential equations.

(8) Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} t^2+3 \\ 2t \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t^3 \\ 3t^2 \end{pmatrix}$.

(a) Compute the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2](t)$.

(b) Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x}$ wherever $W[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$.

(9) Consider two interconnected tanks filled with brine (salt water). The first tank contains 100 liters and the second contains 50 liters. Brine flows with a concentration of 2 grams of salt per liter flows into the first tank at a rate of 3 liters per hour. Well stirred brine flows from the first tank to the second at a rate of 5 liters per hour, from the second to the first at a rate of 2 liters per hour, and from the second into a drain at a rate of 3 liters per hour. At $t = 0$ there are 5 grams of salt in the first tank and 20 grams in the second. Give an initial-value problem that governs the amount of salt in each tank as a function of time.

(10) Consider the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - xy \\ 4y - xy - 2y^2 \end{pmatrix} .$$

(a) Find all of its stationary point.

(b) Compute the coefficient matrix of the linearization associated with each stationary point.

(11) Suppose you know that for some first-order planar system of nonlinear ordinary differential equations:

- its stationary points are $(0,0)$, $(4,-2)$, and $(4,2)$;
- for $(0,0)$ the coefficient matrix of the linearization has eigenvalues -2 and -1 with respective eigenvectors
$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} ;$$
- for $(4,-2)$ the coefficient matrix of the linearization has eigenvalues 2 and 1 with respective eigenvectors
$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} ;$$
- for $(4,2)$ the coefficient matrix of the linearization has eigenvalues 1 and -1 with respective eigenvectors
$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

Sketch a plausible phase portrait for the system. Identify the type and stability of each stationary point.