## Sample Problems for Third In-Class Exam
## Math 246, Spring 2008, Professor David Levermore

(1) Consider the matrices

$$\mathbf{A} = \begin{pmatrix} -i2 & 1+i \\ 2+i & -4 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix}.$$

Compute the matrices

(a) $\mathbf{A}^T$ ,

**Solution.** The transpose of $\mathbf{A}$ is

$$\mathbf{A}^T = \begin{pmatrix} -i2 & 2+i \\ 1+i & -4 \end{pmatrix}.$$

(b) $\overline{\mathbf{A}}$ ,

**Solution.** The conjugate of $\mathbf{A}$ is

$$\overline{\mathbf{A}} = \begin{pmatrix} i2 & 1-i \\ 2-i & -4 \end{pmatrix}.$$

(c) $\mathbf{A}^*$ ,

**Solution.** The adjoint of $\mathbf{A}$ is

$$\mathbf{A}^* = \begin{pmatrix} i2 & 2-i \\ 1-i & -4 \end{pmatrix}.$$

(d) $5\mathbf{A} - \mathbf{B}$ ,

**Solution.** The difference of $5\mathbf{A}$ and $\mathbf{B}$ is given by

$$5\mathbf{A} - \mathbf{B} = \begin{pmatrix} -i10 & 5+i5 \\ 10+i5 & -20 \end{pmatrix} - \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix} = \begin{pmatrix} -7-i10 & -1+i5 \\ 2+i5 & -27 \end{pmatrix}.$$

(e) $\mathbf{AB}$ ,

**Solution.** The product of $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\begin{aligned}\mathbf{AB} &= \begin{pmatrix} -i2 & 1+i \\ 2+i & -4 \end{pmatrix}\begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix} \\ &= \begin{pmatrix} -i2\cdot 7 + (1+i)\cdot 8 & -i2\cdot 6 + (1+i)\cdot 7 \\ (2+i)\cdot 7 - 4\cdot 8 & (2+i)\cdot 6 - 4\cdot 7 \end{pmatrix} \\ &= \begin{pmatrix} 8-i6 & 7-i5 \\ -18+i7 & -16+i6 \end{pmatrix}.\end{aligned}$$

(f) $\mathbf{B}^{-1}$ .

**Solution.** Observe that it is clear that $\mathbf{B}$ has an inverse because

$$\det(\mathbf{B}) = \det\begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix} = 7\cdot 7 - 6\cdot 8 = 49 - 48 = 1\,.$$

The inverse of $\mathbf{B}$ is given by

$$\mathbf{B}^{-1} = \frac{1}{\det(\mathbf{B})}\begin{pmatrix} 7 & -6 \\ -8 & 7 \end{pmatrix} = \begin{pmatrix} 7 & -6 \\ -8 & 7 \end{pmatrix}\,.$$

(2) Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 3 \\ 4 & -1 \end{pmatrix}\,.$$

(a) Find all the eigenvalues of $\mathbf{A}$.

**Solution.** The characteristic polynomial of $\mathbf{A}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z - 15 = (z-1)^2 - 16\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $1\pm 4$, or simply $-3$ and 5.

(b) For each eigenvalue of $\mathbf{A}$ find all of its eigenvectors.

**Solution (using the Cayley-Hamilton method from notes).** One has

$$\mathbf{A} + 3\mathbf{I} = \begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix}\,, \qquad \mathbf{A} - 5\mathbf{I} = \begin{pmatrix} -2 & 3 \\ 4 & -6 \end{pmatrix}\,.$$

Every nonzero column of $\mathbf{A} - 5\mathbf{I}$ has the form

$$\alpha_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{for some } \alpha_1 \neq 0\,.$$

These are all the eigenvectors associated with $-3$. Similarly, every nonzero column of $\mathbf{A} + 3\mathbf{I}$ has the form

$$\alpha_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \text{for some } \alpha_2 \neq 0\,.$$

These are all the eigenvectors associated with 5.

(3) Solve each of the following initial-value problems.

(a) $$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\,, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}\,.$$

**Solution.** The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - z - 12 = (z+3)(z-4)\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $-3$ and 4. These have the form $\frac{1}{2} \pm \frac{7}{2}$. One therefore has

$$\begin{aligned} e^{t\mathbf{A}} &= e^{\frac{1}{2}t}\left[\mathbf{I}\cosh\left(\tfrac{7}{2}t\right) + \left(\mathbf{A} - \tfrac{1}{2}\mathbf{I}\right)\frac{\sinh\left(\frac{7}{2}t\right)}{\frac{7}{2}}\right] \\ &= e^{\frac{1}{2}t}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cosh\left(\tfrac{7}{2}t\right) + \begin{pmatrix} \frac{3}{2} & 2 \\ 5 & -\frac{3}{2} \end{pmatrix}\frac{\sinh\left(\frac{7}{2}t\right)}{\frac{7}{2}}\right] \\ &= e^{\frac{1}{2}t}\begin{pmatrix} \cosh\left(\frac{7}{2}t\right) + \frac{3}{7}\sinh\left(\frac{7}{2}t\right) & \frac{4}{7}\sinh\left(\frac{7}{2}t\right) \\ \frac{10}{7}\sinh\left(\frac{7}{2}t\right) & \cosh\left(\frac{7}{2}t\right) - \frac{3}{7}\sinh\left(\frac{7}{2}t\right) \end{pmatrix}. \end{aligned}$$

The solution of the initial-value problem is therefore

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}}\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = e^{t\mathbf{A}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= e^{\frac{1}{2}t}\begin{pmatrix} \cosh\left(\frac{7}{2}t\right) + \frac{3}{7}\sinh\left(\frac{7}{2}t\right) & \frac{4}{7}\sinh\left(\frac{7}{2}t\right) \\ \frac{10}{7}\sinh\left(\frac{7}{2}t\right) & \cosh\left(\frac{7}{2}t\right) - \frac{3}{7}\sinh\left(\frac{7}{2}t\right) \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= e^{\frac{1}{2}t}\begin{pmatrix} \cosh\left(\frac{7}{2}t\right) - \frac{1}{7}\sinh\left(\frac{7}{2}t\right) \\ -\cosh\left(\frac{7}{2}t\right) + \frac{13}{7}\sinh\left(\frac{7}{2}t\right) \end{pmatrix}. \end{aligned}$$

(b) $$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

**Solution.** The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z + 5 = (z-1)^2 + 4.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $1 \pm i2$. One therefore has

$$\begin{aligned} e^{t\mathbf{A}} &= e^{t}\left[\mathbf{I}\cos(2t) + \left(\mathbf{A} - \mathbf{I}\right)\frac{\sin(2t)}{2}\right] \\ &= e^{t}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(2t) + \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\frac{\sin(2t)}{2}\right] \\ &= e^{t}\begin{pmatrix} \cos(2t) & \frac{1}{2}\sin(2t) \\ -2\sin(2t) & \cos(2t) \end{pmatrix}. \end{aligned}$$

The solution of the initial-value problem is therefore

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}}\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = e^{t\mathbf{A}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= e^{t}\begin{pmatrix} \cos(2t) & \frac{1}{2}\sin(2t) \\ -2\sin(2t) & \cos(2t) \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= e^{t}\begin{pmatrix} \cos(2t) + \frac{1}{2}\sin(2t) \\ -2\sin(2t) + \cos(2t) \end{pmatrix}. \end{aligned}$$

(4) Compute $e^{t\mathbf{A}}$ for $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$.

**Solution.** The characteristic polynomial of $\mathbf{A}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z - 3 = (z-1)^2 - 4\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $1 \pm 2$. One then has

$$\begin{aligned} e^{t\mathbf{A}} &= e^t \left[ \mathbf{I}\cosh(2t) + \big(\mathbf{A} - \mathbf{I}\big)\frac{\sinh(2t)}{2} \right] \\ &= e^t \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cosh(2t) + \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix} \frac{\sinh(2t)}{2} \right] \\ &= e^t \begin{pmatrix} \cosh(2t) & 2\sinh(2t) \\ \frac{1}{2}\sinh(2t) & \cosh(2t) \end{pmatrix}\,. \end{aligned}$$

(5) Find a general solution for each of the following systems.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

**Solution.** The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z + 1 = (z-1)^2\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which is 1, a double root. One therefore has

$$\begin{aligned} e^{t\mathbf{A}} &= e^t \big[\mathbf{I} + \big(\mathbf{A} - \mathbf{I}\big)t\big] = e^t \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} t \right] \\ &= e^t \begin{pmatrix} 1+2t & -4t \\ t & 1-2t \end{pmatrix}\,. \end{aligned}$$

A general solution is therefore given by

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = e^t \begin{pmatrix} 1+2t & -4t \\ t & 1-2t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= e^t \begin{pmatrix} c_1(1+2t) - 4c_2 t \\ c_1 t + c_2(1-2t) \end{pmatrix}\,. \end{aligned}$$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

**Solution.** The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 16\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $\pm i4$. One therefore has

$$\begin{aligned} e^{t\mathbf{A}} &= \left[\mathbf{I}\cos(4t) + \mathbf{A}\frac{\sin(4t)}{4}\right] = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(4t) + \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}\frac{\sin(4t)}{4}\right] \\ &= \begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) & -\frac{5}{4}\sin(4t) \\ \sin(4t) & \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix}. \end{aligned}$$

A general solution is therefore given by

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) & -\frac{5}{4}\sin(4t) \\ \sin(4t) & \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= \begin{pmatrix} c_1\big(\cos(4t) + \frac{1}{2}\sin(4t)\big) - c_2\frac{5}{4}\sin(4t) \\ c_1\sin(4t) + c_2\big(\cos(4t) - \frac{1}{2}\sin(4t)\big) \end{pmatrix}. \end{aligned}$$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

**Solution.** The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 5 & 4 \\ -5 & 1 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 6z + 25 = (z-3)^2 + 16\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $3 \pm i4$. One therefore has

$$\begin{aligned} e^{t\mathbf{A}} &= e^{3t}\left[\mathbf{I}\cos(4t) + (\mathbf{A} - 3\mathbf{I})\frac{\sin(4t)}{4}\right] \\ &= e^{3t}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(4t) + \begin{pmatrix} 2 & 4 \\ -5 & -2 \end{pmatrix}\frac{\sin(4t)}{4}\right] \\ &= e^{3t}\begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) & \sin(4t) \\ -\frac{5}{4}\sin(4t) & \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix}. \end{aligned}$$

A general solution is therefore given by

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = e^{3t}\begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) & \sin(4t) \\ -\frac{5}{4}\sin(4t) & \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= e^{3t}\begin{pmatrix} c_1\big(\cos(4t) + \frac{1}{2}\sin(4t)\big) + c_2\sin(4t) \\ -c_1\frac{5}{4}\sin(4t) + c_2\big(\cos(4t) - \frac{1}{2}\sin(4t)\big) \end{pmatrix}. \end{aligned}$$

(6) Sketch the phase portrait for each of the systems in Problem 5.

(a) **Solution.** Because the characteristic polynomial of $\mathbf{A}$ is $p(z) = (z-1)^2$, one sees that $\mu = 1$ and $\delta = 0$. Because

$$\mathbf{A} - \mathbf{I} = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix},$$

we see that the eigenvectors associated with 1 are

$$\alpha \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{for some } \alpha \neq 0\,.$$

Because $\mu = 1 > 0$, $\delta = 0$, and $a_{21} > 0$ the phase portrait is a *counterclockwise twist source*. The origin is thereby *unstable*. The phase portrait should show there is one trajectory that emerges from the origin on each side of the line $y = x/2$. Every other trajectory emerges from the origin with a clockwise twist.

(b) **Solution.** Because the characteristic polynomial of $\mathbf{A}$ is $p(z) = z^2 + 16$, one sees that $\mu = 0$ and $\delta = -16$. There are no real eigenpairs. Because $\mu = 0$, $\delta = -16 < 0$, and $a_{21} > 0$ the phase portrait is a *counterclockwise center*. The origin is thereby *stable*. The phase portrait should indicate a family of counterclockwise elliptical trajectories that go around the origin.

(c) **Solution.** Because the characteristic polynomial of $\mathbf{A}$ is $p(z) = (z-3)^2 + 16$, one sees that $\mu = 3$ and $\delta = -16$. There are no real eigenpairs. Because $\mu = 3$, $\delta = -16 < 0$, and $a_{21} < 0$ the phase portrait is a *clockwise spiral source*. The origin is thereby *unstable*. The phase portrait should indicate a family of clockwise spiral trajectories that emerge from the origin.

(7) Transform the equation $\dfrac{\mathrm{d}^3u}{\mathrm{d}t^3} + t^2\dfrac{\mathrm{d}u}{\mathrm{d}t} - 3u = \sinh(2t)$ into a first-order system of ordinary differential equations.

**Solution:** Because the equation is third order, the first order system must have dimension three. The simplest such first order system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \sinh(2t) + 3x_1 - t^2x_2 \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} u \\ u' \\ u'' \end{pmatrix}.$$

(8) Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} t^2+3 \\ 2t \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t^3 \\ 3t^2 \end{pmatrix}$.

(a) Compute the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2](t)$.

**Solution.**

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \det\begin{pmatrix} t^2+3 & t^3 \\ 2t & 3t^2 \end{pmatrix} = 3t^4 + 9t^2 - 2t^4 = t^4 + 9t^2 \,.$$

(b) Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x}$ wherever $W[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$.

**Solution.** Let $\mathbf{\Psi}(t) = \begin{pmatrix} t^2+3 & t^3 \\ 2t & 3t^2 \end{pmatrix}$. Because $\dfrac{\mathbf{\Psi}(t)}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{\Psi}(t)$, one has

$$\begin{aligned}
\mathbf{A}(t) &= \frac{\mathbf{\Psi}(t)}{\mathrm{d}t}\mathbf{\Psi}(t)^{-1} = \begin{pmatrix} 2t & 3t^2 \\ 2 & 6t \end{pmatrix} \begin{pmatrix} t^2+3 & t^3 \\ 2t & 3t^2 \end{pmatrix}^{-1} \\
&= \frac{1}{t^4+9t^2} \begin{pmatrix} 2t & 3t^2 \\ 2 & 6t \end{pmatrix} \begin{pmatrix} 3t^2 & -t^3 \\ -2t & t^2+3 \end{pmatrix} = \frac{1}{t^4+9t^2} \begin{pmatrix} 0 & t^4+9t^2 \\ -6t^2 & 4t^3+18t \end{pmatrix} \,.
\end{aligned}$$

(9) Consider two interconnected tanks filled with brine (salt water). The first tank contains 100 liters and the second contains 50 liters. Brine flows with a concentration of 2 grams of salt per liter flows into the first tank at a rate of 3 liters per hour. Well stirred brine flows from the first tank to the second at a rate of 5 liters per hour, from the second to the first at a rate of 2 liters per hour, and from the second into a drain at a rate of 3 liters per hour. At $t = 0$ there are 5 grams of salt in the first tank and 20 grams in the second. Give an initial-value problem that governs the amount of salt in each tank as a function of time.

**Solution:** The rates work out so there will always be 100 liters of brine in the first tank and 50 liters in the second. Let $S_1(t)$ be the grams of salt in the first tank and $S_2(t)$ be the grams of salt in the second tank. These are governed by the initial-value problem

$$\begin{aligned}
\frac{\mathrm{d}S_1}{\mathrm{d}t} &= 2\cdot 3 + \frac{S_2}{50}\,2 - \frac{S_1}{100}\,5 \,, & S_1(0) &= 2 \,, \\
\frac{\mathrm{d}S_2}{\mathrm{d}t} &= \frac{S_1}{100}\,5 - \frac{S_2}{50}\,2 - \frac{S_2}{50}\,3 \,, & S_2(0) &= 20 \,.
\end{aligned}$$

You could leave the answer in the above form. It can however be simplified to

$$\begin{aligned}
\frac{\mathrm{d}S_1}{\mathrm{d}t} &= 6 + \frac{S_2}{25} - \frac{S_1}{20} \,, & S_1(0) &= 2 \,, \\
\frac{\mathrm{d}S_2}{\mathrm{d}t} &= \frac{S_1}{20} - \frac{S_2}{10} \,, & S_2(0) &= 20 \,.
\end{aligned}$$

(10) Consider the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - xy \\ 4y - xy - 2y^2 \end{pmatrix} .$$

(a) Find all of its stationary point.

**Solution.** Stationary points satisfy

$$0 = x - xy = x(1 - y) ,$$
$$0 = 4y - xy - 2y^2 = y(4 - x - 2y) .$$

The top equation above shows that either $x = 0$ or $y = 1$. When $x = 0$ the bottom equation shows that

$$0 = y(4 - 2y) ,$$

whereby either $y = 0$ or $y = 2$. Hence, $(0, 0)$ and $(0, 2)$ are stationary points. When $y = 1$ the bottom equation shows that

$$0 = 1(4 - x - 2) = 2 - x ,$$

whereby $x = 2$. Hence, $(2, 1)$ is a stationary point. All the stationary points of the system are therefore

$$(0, 0) , \qquad (0, 2) , \qquad (2, 1) .$$

(b) Compute the coefficient matrix of the linearization associated with each stationary point.

**Solution.** Because

$$\begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix} = \begin{pmatrix} x - xy \\ 4y - xy - 2y^2 \end{pmatrix} ,$$

the matrix of partial derivatives is

$$\begin{pmatrix} \partial_x f(x, y) & \partial_y f(x, y) \\ \partial_x g(x, y) & \partial_y g(x, y) \end{pmatrix} = \begin{pmatrix} 1 - y & -x \\ -y & 4 - x - 4y \end{pmatrix} .$$

By evaluating this matrix at each of the stationary points, you find the coefficient matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \qquad \text{at } (0, 0) ,$$
$$\mathbf{A} = \begin{pmatrix} -1 & 0 \\ -2 & -4 \end{pmatrix} \qquad \text{at } (0, 2) ,$$
$$\mathbf{A} = \begin{pmatrix} 0 & -2 \\ -1 & -2 \end{pmatrix} \qquad \text{at } (2, 1) ,$$

This is all you were asked to do. However, if you had been asked to classify the type and stability of each stationary point then you can easily see that $(0, 0)$ is a nodal source (the eigenvalues are 1 and 4) and is thereby unstable, $(0, 2)$ is a nodal sink (the eigenvalues are $-1$ and $-4$) and is thereby asymptotically stable, while $(2, 1)$ is a saddle (the eigenvalues are $-1 - \sqrt{3}$ and $-1 + \sqrt{3}$) and is thereby unstable.

(11) Suppose you know that for some first-order planar system of nonlinear ordinary differential equations:

- its stationary points are $(0,0)$, $(4,-2)$, and $(4,2)$;
- for $(0,0)$ the coefficient matrix of the linearization has eigenvalues $-2$ and $-1$ with respective eigenvectors

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix} ;$$

- for $(4,-2)$ the coefficient matrix of the linearization has eigenvalues 2 and 1 with respective eigenvectors

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} ;$$

- for $(4,2)$ the coefficient matrix of the linearization has eigenvalues 1 and $-1$ with respective eigenvectors

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

Sketch a plausible phase portrait for the system. Identify the type and stability of each stationary point.

**Solution.** The type and stability of each stationary point are determined as follows.

- The stationary point $(0,0)$ has two negative simple real eigenvalues. It therefore is a *nodal sink* and thereby is *asymptotically stable*.
- The stationary point $(4,-2)$ has two positive simple real eigenvalues. It therefore is a *nodal source* and thereby is *unstable*.
- The stationary point $(4,2)$ has one negative and one positive real eigenvalue. It therefore is a *saddle* and thereby is *unstable*.

There are many plausible phase portraits that one might draw. One was given in the review session. They all however have the following features.

- Near the nodal sink $(0,0)$ there is one trajectory that approaches $(0,0)$ tangent to each side of the line $y=-x$. Every other trajectory approaches $(0,0)$ tangent to the line $y=x$.
- Near the nodal source $(4,-2)$ there is one trajectory that emerges from $(4,-2)$ tangent to each side of the line $y=-x+2$. Every other trajectory emerges from $(4,-2)$ tangent to the line $x=4$.
- Near the saddle $(4,2)$ there is one trajectory that emerges from $(4,2)$ tangent to each side of the line $y=x-2$. There is also one trajectory that approaches $(4,2)$ tangent to each side of the line $x=4$.