Sample Final Exam Problems, Math 246, Spring 2008

(1) Consider the differential equation $\frac{dy}{dt} = y^2(4 - y^2)$ over the interval $-5 \leq y \leq 5$.

(a) Sketch its phase-line portrait in this interval.
(b) Identify its equilibrium (stationary) points and discuss their stability.
(c) If $y(0) = 3$, how does the solution $y(t)$ behave as $t \to \infty$?

(2) Solve (possibly implicitly) each of the following initial-value problems.

(a) $\frac{dy}{dt} + \frac{2ty}{1+t^2} = t^2\,, \quad y(0) = 1\,.$
(b) $\frac{dy}{dx} + \frac{e^x y + 2x}{2y + e^x} = 0\,, \quad y(0) = 0\,.$

(3) Let $y(t)$ be the solution of the initial-value problem

$$\frac{dy}{dt} = y^2 + t^2\,, \qquad y(0) = 1\,.$$

Use one step of the improved Euler (trapezoidal-Heun) method to approximate $y(0.1)$.

(4) Give an explicit general solution of the following equations.

(a) $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = te^t + \cos(2t)$
(b) $\frac{d^2y}{dt^2} + y = \tan(t)$

(5) When a mass of 3 kilograms is hung vertically from a spring, it stretches the spring 0.25 meters. (Gravitational acceleration is 9.8 m/sec^2.) At $t = 0$ the mass is set in motion from 0.5 meters below its equilibrium (rest) position with a upward velocity of 2 m/sec. Neglect drag and assume that the spring force is proportional to its displacement. Formulate an initial-value problem that governs the motion of the mass for $t > 0$. (DO NOT solve this initial-value problem; just write it down!)

(6) Give an explicit general solution of the equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 10y = 0\,.$$

Sketch a typical solution. If this equation governs a damped spring-mass system, is the system over, under. or critically damped?

(7) Find the Laplace transform $Y(s)$ of the solution $y(t)$ to the initial-value problem

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 8y = g(t)\,, \qquad y(0) = 2\,, \quad y'(0) = 4\,.$$

where

$$g(t) = \begin{cases} 4 & \text{for } 0 \leq t < 2\,, \\ t^2 & \text{for } 2 \leq t\,. \end{cases}$$

You may refer to the table in Section 6.2 of the book. (DO NOT take the inverse Laplace transform to find $y(t)$; just solve for $Y(s)$!)

(8) Find the function $y(t)$ whose Laplace transform $Y(s)$ is given by $Y(s) = \dfrac{e^{-3s}4}{s^2 - 6s + 5}$. You may refer to the table in Section 6.2 of the book.

(9) Give a general solution of $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x}$ for the following $\mathbf{A}$.

(a) $\mathbf{A} = \begin{pmatrix} 6 & 4 \\ 4 & 0 \end{pmatrix}$

(b) $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$

(10) A matrix $\mathbf{A}$ has eigenvalues -2 and -1 with associated eigenvectors $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

(a) Give a general solution to $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x}$.

(b) Sketch a phase-plane portrait for this system. Identify its type. (Carefully mark the exact lines, curves, and arrows!)

(11) Consider the nonlinear system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -5y\,,$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = x - 4y - x^2\,.$$

(a) Find all of its equilibrium (critical, stationary) points.

(b) Compute the coefficient matrix of the linearization (the derivative matrix) at each equilibrium (critical, stationary) point.

(c) Identify the type and stability of each equilibrium (critical, stationary) point.

(d) Sketch a plausible global phase-plane portrait. (Carefully mark the exact lines, curves, and arrows!)

(12) Consider the nonlinear system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(3 - 3x + 2y)\,,$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = y(6 - x - y)\,.$$

(a) Find all of its equilibrium (critical, stationary) points.

(b) Compute the coefficient matrix of the linearization (the derivative matrix) at each equilibrium (critical, stationary) point.

(c) Identify the type and stability of each equilibrium (critical, stationary) point.

(d) Sketch a plausible global phase-plane portrait. (Carefully mark the exact lines, curves, and arrows!)