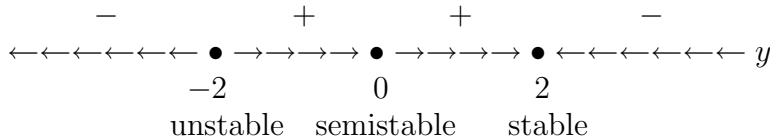


Solutions to Sample Final Exam Problems, Math 246, Spring 2008

- (1) Consider the differential equation $\frac{dy}{dt} = y^2(4 - y^2)$ over the interval $-5 \leq y \leq 5$.
- (a) Sketch its phase-line portrait in this interval.
 - (b) Identify its equilibrium (stationary) points and discuss their stability.
 - (c) If $y(0) = 3$, how does the solution $y(t)$ behave as $t \rightarrow \infty$?

Solution (a,b): The right-hand side factors as $y^2(2 + y)(2 - y)$. The stationary solutions are $y = -2$, $y = 0$, and $y = 2$. A sign analysis of $y^2(2 + y)(2 - y)$ shows that the phase-line portrait for this equation is therefore



Solution (c): The phase-line shows that if $y(0) = 3$ then $y(t) \rightarrow 2$ as $t \rightarrow \infty$.

- (2) Solve (possibly implicitly) each of the following initial-value problems.

(a) $\frac{dy}{dt} + \frac{2ty}{1 + t^2} = t^2$, $y(0) = 1$.

Solution: This equation is *linear* and is already in normal form. An integrating factor is

$$\exp\left(\int_0^t \frac{2s}{1 + s^2} ds\right) = \exp(\log(1 + t^2)) = 1 + t^2,$$

so that

$$\frac{d}{dt}((1 + t^2)y) = (1 + t^2)t^2 = t^2 + t^4.$$

Integrate this to obtain

$$(1 + t^2)y = \frac{1}{3}t^3 + \frac{1}{5}t^5 + C.$$

The initial condition $y(0) = 1$ implies that $C = (1 + 0^2) \cdot 1 - \frac{1}{3}0^3 - \frac{1}{5}0^5 = 1$. Therefore

$$y = \frac{1 + \frac{1}{3}t^3 + \frac{1}{5}t^5}{1 + t^2}.$$

(b) $\frac{dy}{dx} + \frac{e^x y + 2x}{2y + e^x} = 0$, $y(0) = 0$.

Solution: Express this equation in the differential form

$$(e^x y + 2x) dx + (2y + e^x) dy = 0.$$

This differential form is *exact* because

$$\partial_y(e^x y + 2x) = e^x = \partial_x(2y + e^x) = e^x.$$

We can therefore find $H(x, y)$ such that

$$\partial_x H(x, y) = e^x y + 2x, \quad \partial_y H(x, y) = 2y + e^x.$$

The first equation implies $H(x, y) = e^x y + x^2 + h(y)$. Plugging this into the second equation gives

$$e^x + h'(y) = 2y + e^x,$$

which yields $h'(y) = 2y$. Taking $h(y) = y^2$, the general solution is

$$e^x y + x^2 + y^2 = C.$$

The initial condition $y(0) = 0$ implies that $C = e^0 \cdot 0 + 0^2 + 0^2 = 0$. Therefore

$$y^2 + e^x y + x^2 = 0.$$

If you had been asked for an explicit solution then the quadratic formula yields

$$y = \frac{-e^x + \sqrt{e^{2x} - 4x^2}}{2}.$$

Here the positive square root is taken because that solution satisfies the initial condition. It exists wherever $e^{2x} \geq 4x^2$.

(3) Let $y(t)$ be the solution of the initial-value problem

$$\frac{dy}{dt} = y^2 + t^2, \quad y(0) = 1.$$

Use one step of the improved Euler (trapezoidal-Heun) method to approximate $y(0.1)$.

Solution. The improved Euler method is

$$\begin{aligned} f_n &= f(y_n, t_n), \\ \tilde{y}_{n+1} &= y_n + hf_n, \\ t_{n+1} &= t_n + h, \\ \tilde{f}_{n+1} &= f(\tilde{y}_{n+1}, t_{n+1}), \\ y_{n+1} &= y_n + \frac{h}{2}(f_n + \tilde{f}_{n+1}), \end{aligned}$$

where h is the time step, t_0 is the initial time, and y_0 is the initial data.

When the improved Euler method is applied with $h = 0.1$, $t_0 = 0$, $y_0 = 1$, and $f(y, t) = y^2 + t^2$ for one step

$$\begin{aligned} f_0 &= f(y_0, t_0) = y_0^2 + t_0^2 = 1^2 + 0^2 = 1, \\ \tilde{y}_1 &= y_0 + hf_0 = 1 + 0.1 \cdot 1 = 1.1, \\ t_1 &= t_0 + h = 0 + 0.1 = 0.1, \\ \tilde{f}_1 &= f(\tilde{y}_1, t_1) = \tilde{y}_1^2 + t_1^2 = (1.1)^2 + (0.1)^2, \\ y_1 &= y_0 + \frac{h}{2}(f_0 + \tilde{f}_1) = 1 + 0.05(1 + (1.1)^2 + (0.1)^2). \end{aligned}$$

The approximation is therefore

$$y(0.1) \approx y_1 = 1 + 0.05(1 + (1.1)^2 + (0.1)^2).$$

You DO NOT have to work out the arithmetic! If you did then $y_1 = 1.111$.

(4) Give an explicit general solution of the following equations.

(a) $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = te^t + \cos(2t)$

Solution. This is a constant coefficient, inhomogeneous, linear equation. Its characteristic polynomial is

$$P(z) = z^2 - 2z + 5 = (z - 1)^2 + 4 = (z - 1)^2 + 2^2.$$

This has the conjugate pair of roots $1 \pm i2$, which yields a general solution of the associated homogeneous problem

$$y_H(t) = c_1 e^t \cos(2t) + c_2 e^t \sin(2t).$$

A particular solution $y_P(t)$ can be found by the method of undetermined coefficients. The characteristics of the forcing terms te^t and $\cos(2t)$ are $r + is = 1$ and $r + is = i2$ respectively. Because these characteristics are different, they should be treated separately. This can be done using either KEY identity evaluation (as in the lectures) or direct substitution (as in the book).

KEY Identity Evaluations. The forcing term te^t has degree $d = 1$ and characteristic $r + is = 1$, which is a root of $P(z)$ of multiplicity $m = 0$. Because $m + d = 1$, you need the KEY identity and its first derivative

$$\mathbf{L}(e^{zt}) = (z^2 - 2z + 5)e^{zt},$$

$$\mathbf{L}(te^{zt}) = (z^2 - 2z + 5)t e^{zt} + (2z - 2)e^{zt}.$$

Evaluate these at $z = 1$ to find $\mathbf{L}(e^t) = 4e^t$ and $\mathbf{L}(te^t) = 4te^t$. Dividing the second of these equations by 4 yields $\mathbf{L}(\frac{1}{4}te^t) = te^t$, which implies $y_{P1}(t) = \frac{1}{4}te^t$. The forcing term $\cos(2t)$ has degree $d = 0$ and characteristic $r + is = i2$, which is a root of $P(z)$ of multiplicity $m = 0$. Because $m + d = 0$, you only need the KEY identity

$$\mathbf{L}(e^{zt}) = (z^2 - 2z + 5)e^{zt}.$$

Evaluate this at $z = i2$ to find $\mathbf{L}(e^{i2t}) = (1 - i4)e^{i2t}$. Dividing this by $(1 - i4)$ yields

$$\mathbf{L}\left(\frac{e^{i2t}}{1 - i4}\right) = e^{i2t}.$$

Because $\cos(2t) = \operatorname{Re}(e^{i2t})$, the above equation implies

$$\begin{aligned} y_{P2}(t) &= \operatorname{Re}\left(\frac{e^{i2t}}{1 - i4}\right) = \operatorname{Re}\left(\frac{(1 + i4)e^{i2t}}{1^2 + 4^2}\right) \\ &= \frac{1}{17} \operatorname{Re}((1 + i4)e^{i2t}) = \frac{1}{17}(\cos(2t) - 4\sin(2t)). \end{aligned}$$

Combining these particular solutions with the general solution of the associated homogeneous problem yields the general solution

$$\begin{aligned} y &= y_H(t) + y_{P1}(t) + y_{P2}(t) \\ &= c_1 e^t \cos(2t) + c_2 e^t \sin(2t) + \frac{1}{4}te^t + \frac{1}{17}\cos(2t) - \frac{4}{17}\sin(2t). \end{aligned}$$

Direct Substitution. The forcing term $t e^t$ has degree $d = 1$ and characteristic $r + is = 1$, which is a root of $P(z)$ of multiplicity $m = 0$. Because $m = 0$ and $m + d = 1$, you seek a particular solution of the form

$$y_{P_1}(t) = A_0 t e^t + A_1 e^t.$$

Because

$$y'_{P_1}(t) = A_0 t e^t + (A_0 + A_1) e^t, \quad y''_{P_1}(t) = A_0 t e^t + (2A_0 + A_1) e^t,$$

one sees that

$$\begin{aligned} \mathbf{L}y_{P_1}(t) &= y''_{P_1}(t) - 2y'_{P_1}(t) + 5y_{P_1}(t) \\ &= (A_0 t e^t + (2A_0 + A_1) e^t) - 2(A_0 t e^t + (A_0 + A_1) e^t) \\ &\quad + 5(A_0 t e^t + A_1 e^t) \\ &= 4A_0 t e^t + 4A_1 e^t. \end{aligned}$$

Setting $4A_0 t e^t + 4A_1 e^t = t e^t$, we see that $4A_0 = 1$ and $4A_1 = 0$, whereby $A_0 = \frac{1}{4}$ and $A_1 = 0$. Hence, $y_P(t) = \frac{1}{4} t e^t$.

The forcing term $\cos(2t)$ has degree $d = 0$ and characteristic $r + is = i2$, which is a root of $P(z)$ of multiplicity $m = 0$. Because $m = 0$ and $m + d = 0$, you seek a particular solution of the form

$$y_{P_2}(t) = A \cos(2t) + B \sin(2t).$$

Because

$$\begin{aligned} y'_{P_2}(t) &= -2A \sin(2t) + 2B \cos(2t), \\ y''_{P_2}(t) &= -4A \cos(2t) - 4B \sin(2t), \end{aligned}$$

one sees that

$$\begin{aligned} \mathbf{L}y_{P_2}(t) &= y''_{P_2}(t) - 2y'_{P_2}(t) + 5y_{P_2}(t) \\ &= (-4A \cos(2t) - 4B \sin(2t)) - 2(-2A \sin(2t) + 2B \cos(2t)) \\ &\quad + 5(A \cos(2t) + B \sin(2t)) \\ &= (A - 4B) \cos(2t) + (B + 4A) \sin(2t). \end{aligned}$$

Setting $(A - 4B) \cos(2t) + (B + 4A) \sin(2t) = \cos(2t)$, we see that

$$A - 4B = 1, \quad B + 4A = 0.$$

This system can be solved by any method you choose to find $A = \frac{1}{17}$ and $B = -\frac{4}{17}$, whereby

$$y_{P_2}(t) = \frac{1}{17} \cos(2t) - \frac{4}{17} \sin(2t).$$

Combining these particular solutions with the general solution of the associated homogeneous problem yields the general solution

$$\begin{aligned} y &= y_H(t) + y_{P_1}(t) + y_{P_2}(t) \\ &= c_1 e^t \cos(2t) + c_2 e^t \sin(2t) + \frac{1}{4} t e^t + \frac{1}{17} \cos(2t) - \frac{4}{17} \sin(2t). \end{aligned}$$

(b) $\frac{d^2y}{dt^2} + y = \tan(t)$

Solution. This is a constant coefficient, inhomogeneous, linear equation. Its characteristic polynomial is

$$P(z) = z^2 + 1 = z^2 + 1^2.$$

This has the conjugate pair of roots $\pm i$, which yields a general solution of the associated homogeneous problem

$$y_H(t) = c_1 \cos(t) + c_2 \sin(t).$$

Because of the form of the forcing term, you must use the method of variation of parameters to find a particular solution. The equation is already in normal form. Seek a solution in the form

$$y_H(t) = u_1(t) \cos(t) + u_2(t) \sin(t),$$

where $u_1'(t)$ and $u_2'(t)$ satisfy

$$\begin{aligned} u_1'(t) \cos(t) + u_2'(t) \sin(t) &= 0, \\ -u_1'(t) \sin(t) + u_2'(t) \cos(t) &= \tan(t). \end{aligned}$$

Solve this system to find

$$u_1'(t) = -\frac{\sin(t)^2}{\cos(t)} = \cos(t) - \sec(t), \quad u_2'(t) = \sin(t).$$

Integrate these to find

$$u_1(t) = c_1 + \sin(t) - \log(\tan(t) + \sec(t)), \quad u_2(t) = c_2 - \cos(t).$$

A general solution is therefore

$$y = c_1 \cos(t) + c_2 \sin(t) - \cos(t) \log(\tan(t) + \sec(t)).$$

- (5) When a mass of 3 kilograms is hung vertically from a spring, it stretches the spring 0.25 meters. (Gravitational acceleration is 9.8 m/sec².) At $t = 0$ the mass is set in motion from 0.5 meters below its equilibrium (rest) position with a upward velocity of 2 m/sec. Neglect drag and assume that the spring force is proportional to its displacement. Formulate an initial-value problem that governs the motion of the mass for $t > 0$. (DO NOT solve this initial-value problem; just write it down!)

Solution. Let $h(t)$ be the displacement (in meters) of the mass from its equilibrium (rest) position at time t (in seconds), with upward displacements being positive. The governing initial-value problem then has the form

$$m \frac{d^2h}{dt^2} + kh = 0, \quad h(0) = -.5, \quad h'(0) = 2,$$

where m is the mass and k is the spring constant. The problem says that $m = 3$ kilograms. The spring constant is obtained by balancing the weight of the mass ($mg = 3 \cdot 9.8$ Newtons) with the force applied by the spring when it is stretched .25 m. This gives $k \cdot 25 = 3 \cdot 9.8$, or

$$k = \frac{3 \cdot 9.8}{.25} = 12 \cdot 9.8 \text{ Newtons/m.}$$

The governing initial-value problem is therefore

$$3\frac{d^2h}{dt^2} + 12 \cdot 9.8h = 0, \quad h(0) = -.5, \quad h'(0) = 2.$$

Had you chosen positive h to be downward displacements then the only thing that would differ is the sign of the initial data.

- (6) Give an explicit general solution of the equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 10y = 0.$$

Sketch a typical solution. If this equation governs a damped spring-mass system, is the system over, under, or critically damped?

Solution. This is a constant coefficient, inhomogeneous, linear equation. Its characteristic polynomial is

$$P(z) = z^2 + 2z + 10 = (z + 1)^2 + 3^2.$$

This has the conjugate pair of roots $-1 \pm i3$, which yields a general solution

$$y = c_1e^{-t} \cos(3t) + c_2e^{-t} \sin(3t).$$

When $c_1^2 + c_2^2 > 0$ this can be put into the amplitude-phase form

$$y = Ae^{-t} \cos(3t - \delta),$$

where $A > 0$ and $0 \leq \delta < 2\pi$ are determined from c_1 and c_2 by

$$A = \sqrt{c_1^2 + c_2^2}, \quad \cos(\delta) = \frac{c_1}{A}, \quad \sin(\delta) = \frac{c_2}{A}.$$

The sketch should show a decaying oscillation with amplitude Ae^{-t} and quasiperiod $\frac{2\pi}{3}$. This corresponds to an *under damped* spring-mass system.

- (7) Find the Laplace transform $Y(s)$ of the solution $y(t)$ to the initial-value problem

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 8y = g(t), \quad y(0) = 2, \quad y'(0) = 4.$$

where

$$g(t) = \begin{cases} 4 & \text{for } 0 \leq t < 2, \\ t^2 & \text{for } 2 \leq t. \end{cases}$$

You may refer to the table in Section 6.2 of the book. (DO NOT take the inverse Laplace transform to find $y(t)$; just solve for $Y(s)$!)

Solution. The Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) + 4\mathcal{L}[y'](s) + 8\mathcal{L}[y](s) = \mathcal{L}[g](s),$$

where

$$\begin{aligned} \mathcal{L}[y](s) &= Y(s), \\ \mathcal{L}[y'](s) &= sY(s) - y(0) = sY(s) - 2, \\ \mathcal{L}[y''](s) &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2s - 4. \end{aligned}$$

To compute $\mathcal{L}[g](s)$, first write g as

$$\begin{aligned} g(t) &= (1 - u(t-2))4 + u(t-2)t^2 = 4 - u(t-2)4 + u(t-2)t^2 \\ &= 4 + u(t-2)(t^2 - 4) = 4 + u(t-2)((2 + (t-2))^2 - 4) \\ &= 4 + u(t-2)(4(t-2) + (t-2)^2). \end{aligned}$$

Referring to the table of Laplace transforms in the book, item 13 with $c = 2$, item 1, and item 3 with $n = 1$ and $n = 2$ then show that

$$\begin{aligned} \mathcal{L}[g](s) &= 4\mathcal{L}[1](s) + 4\mathcal{L}[u(t-2)(t-2)](s) + \mathcal{L}[u(t-2)(t-2)^2](s) \\ &= 4\mathcal{L}[1](s) + 4e^{-2s}\mathcal{L}[t](s) + e^{-2s}\mathcal{L}[t^2](s) \\ &= 4\frac{1}{s} + 4e^{-2s}\frac{1}{s^2} + e^{-2s}\frac{2}{s^3}. \end{aligned}$$

The Laplace transform of the initial-value problem then becomes

$$(s^2Y(s) - 2s - 4) + 4(sY(s) - 2) + 8Y(s) = \frac{4}{s} + e^{-2s}\frac{4}{s^2} + e^{-2s}\frac{2}{s^3},$$

which becomes

$$(s^2 + 4s + 8)Y(s) - 2s - 12 = \frac{4}{s} + e^{-2s}\frac{4}{s^2} + e^{-2s}\frac{2}{s^3}.$$

Hence, $Y(s)$ is given by

$$Y(s) = \frac{1}{s^2 + 4s + 8} \left(2s + 12 + \frac{4}{s} + e^{-2s}\frac{4}{s^2} + e^{-2s}\frac{2}{s^3} \right).$$

- (8) Find the function $y(t)$ whose Laplace transform $Y(s)$ is given by $Y(s) = \frac{e^{-3s}4}{s^2 - 6s + 5}$.

You may refer to the table in Section 6.2 of the book.

Solution. The denominator factors as $(s-5)(s-1)$, so the partial fraction decomposition is

$$\frac{4}{s^2 - 6s + 5} = \frac{4}{(s-5)(s-1)} = \frac{1}{s-5} - \frac{1}{s-1}.$$

Referring to the table of Laplace transforms in the book, item 11 with $n = 0$ and $a = 5$, and with $n = 0$ and $a = 1$ gives

$$\mathcal{L}[e^{5t}](s) = \frac{1}{s-5}, \quad \mathcal{L}[e^t](s) = \frac{1}{s-1},$$

whereby

$$\frac{4}{s^2 - 6s + 5} = \mathcal{L}[e^{5t}](s) - \mathcal{L}[e^t](s) = \mathcal{L}[e^{5t} - e^t](s).$$

It follows from item 13 with $c = 3$ that

$$\mathcal{L}[u(t-3)(e^{5(t-3)} - e^{t-3})](s) = e^{-3s}\frac{4}{s^2 - 6s + 5} = Y(s).$$

You therefore conclude that

$$y(t) = \mathcal{L}^{-1}[Y(s)](t) = u(t-3)(e^{5(t-3)} - e^{t-3}).$$

(9) Give a general solution of $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ for the following \mathbf{A} .

(a) $\mathbf{A} = \begin{pmatrix} 6 & 4 \\ 4 & 0 \end{pmatrix}$

Solution. The characteristic polynomial of \mathbf{A} is

$$\begin{aligned} p(z) &= z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) \\ &= z^2 - 6z - 16 = (z - 3)^2 - 25 = (z - 3)^2 - 5^2. \end{aligned}$$

The eigenvalues of \mathbf{A} are the roots of this polynomial, which are 3 ± 5 , or simply -2 and 8 . One therefore has

$$\begin{aligned} e^{t\mathbf{A}} &= e^{3t} \left[\mathbf{I} \cosh(5t) + (\mathbf{A} - 3\mathbf{I}) \frac{\sinh(5t)}{5} \right] \\ &= e^{3t} \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cosh(5t) + \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \frac{\sinh(5t)}{5} \right] \\ &= e^{3t} \begin{pmatrix} \cosh(5t) + \frac{3}{5} \sinh(5t) & \frac{4}{5} \sinh(5t) \\ \frac{4}{5} \sinh(5t) & \cosh(5t) - \frac{3}{5} \sinh(5t) \end{pmatrix}. \end{aligned}$$

A general solution is therefore given by

$$\mathbf{x} = c_1 e^{3t} \begin{pmatrix} \cosh(5t) + \frac{3}{5} \sinh(5t) \\ \frac{4}{5} \sinh(5t) \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} \frac{4}{5} \sinh(5t) \\ \cosh(5t) - \frac{3}{5} \sinh(5t) \end{pmatrix}.$$

Alternative Solution. The characteristic polynomial of \mathbf{A} is

$$\begin{aligned} p(z) &= z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) \\ &= z^2 - 6z - 16 = (z - 3)^2 - 25 = (z - 3)^2 - 5^2. \end{aligned}$$

The eigenvalues of \mathbf{A} are the roots of this polynomial, which are 3 ± 5 , or simply -2 and 8 . Because

$$\mathbf{A} + 2\mathbf{I} = \begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix}, \quad \mathbf{A} - 8\mathbf{I} = \begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix},$$

we see that \mathbf{A} has the eigenpairs

$$\left(-2, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right), \quad \left(8, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right).$$

A general solution is therefore given by

$$\mathbf{x} = c_1 e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{8t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

$$(b) \mathbf{A} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

Solution. The characteristic polynomial of \mathbf{A} is

$$\begin{aligned} p(z) &= z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) \\ &= z^2 - 2z + 5 = (z - 1)^2 + 4 = (z - 1)^2 + 2^2. \end{aligned}$$

The eigenvalues of \mathbf{A} are the roots of this polynomial, which are $1 \pm i2$. One therefore has

$$\begin{aligned} e^{t\mathbf{A}} &= e^t \left[\mathbf{I} \cos(2t) + (\mathbf{A} - \mathbf{I}) \frac{\sin(2t)}{2} \right] \\ &= e^t \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(2t) + \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \frac{\sin(2t)}{2} \right] \\ &= e^t \begin{pmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{pmatrix}. \end{aligned}$$

A general solution is therefore given by

$$\mathbf{x} = c_1 e^t \begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix}.$$

Alternative Solution. The characteristic polynomial of \mathbf{A} is

$$\begin{aligned} p(z) &= z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) \\ &= z^2 - 2z + 5 = (z - 1)^2 + 4 = (z - 1)^2 + 2^2. \end{aligned}$$

The eigenvalues of \mathbf{A} are the roots of this polynomial, which are $1 \pm i2$. Because

$$\mathbf{A} - (1 + i2)\mathbf{I} = \begin{pmatrix} -i2 & 2 \\ -2 & -i2 \end{pmatrix}, \quad \mathbf{A} - (1 - i2)\mathbf{I} = \begin{pmatrix} i2 & 2 \\ -2 & i2 \end{pmatrix},$$

we see that \mathbf{A} has the eigenpairs

$$\left(1 + i2, \begin{pmatrix} 1 \\ i \end{pmatrix} \right), \quad \left(1 - i2, \begin{pmatrix} -i \\ 1 \end{pmatrix} \right).$$

Because

$$e^{(1+i2)t} \begin{pmatrix} 1 \\ i \end{pmatrix} = e^t \begin{pmatrix} \cos(2t) + i \sin(2t) \\ -\sin(2t) + i \cos(2t) \end{pmatrix},$$

two real solutions of the system are

$$e^t \begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix}, \quad e^t \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix}.$$

A general solution is therefore

$$\mathbf{x} = c_1 e^t \begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix}.$$

(10) A matrix \mathbf{A} has eigenvalues -2 and -1 with associated eigenvectors $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

(a) Give a general solution to $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$.

Solution. A general solution is

$$\mathbf{x} = c_1 e^{-2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

(b) Sketch a phase-plane portrait for this system. Identify its type. (Carefully mark the exact lines, curves, and arrows!)

Solution. The coefficient matrix has two negative eigenvalues. The origin is therefore a *nodal sink* and is thereby *asymptotically stable*. There is one trajectory that approaches $(0,0)$ along each half of the lines $x = 3y$ and $y = -2x$. (These are the lines of eigenvectors.) Every other trajectory approaches $(0,0)$ tangent to the line $y = -2x$, which is the line corresponding to the eigenvalue with the smaller absolute value.

(11) Consider the nonlinear system

$$\begin{aligned} \frac{dx}{dt} &= -5y, \\ \frac{dy}{dt} &= x - 4y - x^2. \end{aligned}$$

(a) Find all of its equilibrium (critical, stationary) points.

Solution. Stationary points satisfy

$$0 = -5y, \quad 0 = x - 4y - x^2.$$

The first equation implies $y = 0$, whereby the second equation becomes $0 = x - x^2 = x(1 - x)$, which implies either $x = 0$ or $x = 1$. All the stationary points of the system are therefore

$$(0, 0), \quad (1, 0).$$

(b) Compute the coefficient matrix of the linearization (the derivative matrix) at each equilibrium (critical, stationary) point.

Solution. Because

$$\begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix} = \begin{pmatrix} -5y \\ x - 4y - x^2 \end{pmatrix},$$

the matrix of partial derivatives is

$$\begin{pmatrix} \partial_x f(x, y) & \partial_y f(x, y) \\ \partial_x g(x, y) & \partial_y g(x, y) \end{pmatrix} = \begin{pmatrix} 0 & -5 \\ 1 - 2x & -4 \end{pmatrix}.$$

Evaluating this matrix at each stationary point yields the coefficient matrices

$$\mathbf{A} = \begin{pmatrix} 0 & -5 \\ 1 & -4 \end{pmatrix} \quad \text{at } (0, 0), \quad \mathbf{A} = \begin{pmatrix} 0 & -5 \\ -1 & -4 \end{pmatrix} \quad \text{at } (1, 0).$$

- (c) Identify the type and stability of each equilibrium (critical, stationary) point.

Solution. The coefficient matrix \mathbf{A} at $(0, 0)$ has eigenvalues that satisfy

$$0 = \det(z\mathbf{I} - \mathbf{A}) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 4z + 5 = (z + 2)^2 + 1^2.$$

The eigenvalues are thereby $-2 \pm i$. Because $a_{21} = 1 > 0$, the stationary point $(0, 0)$ is therefore a *counter clockwise spiral sink*, which is *asymptotically stable*. This is one of the generic types, so it describes the phase-plane portrait of the nonlinear system near $(0, 0)$.

The coefficient matrix \mathbf{A} at $(1, 0)$ has eigenvalues that satisfy

$$0 = \det(z\mathbf{I} - \mathbf{A}) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 4z - 5 = (z + 2)^2 - 3^2.$$

The eigenvalues are thereby -2 ± 3 , or simply -5 and 1 . The stationary point $(1, 0)$ is therefore a *saddle*, which is *unstable*. This is one of the generic types, so it describes the phase-plane portrait of the nonlinear system near $(1, 0)$.

- (d) Sketch a plausible global phase-plane portrait. (Carefully mark the exact lines, curves, and arrows!)

Solution. The stationary point $(0, 0)$ is a *counter clockwise spiral sink*.

The stationary point $(1, 0)$ is a *saddle*. The coefficient matrix \mathbf{A} has eigenvalues -5 and 1 . Because

$$\mathbf{A} + 5\mathbf{I} = \begin{pmatrix} 5 & -5 \\ -1 & 1 \end{pmatrix}, \quad \mathbf{A} - \mathbf{I} = \begin{pmatrix} -1 & -5 \\ -1 & -5 \end{pmatrix},$$

it has the eigenpairs

$$\left(-5, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right), \quad \left(1, \begin{pmatrix} -5 \\ 1 \end{pmatrix}\right)$$

Near $(1, 0)$ there is one trajectory that emerges from $(1, 0)$ tangent to each side of the line $x = 1 - 5y$. There is also one trajectory that approaches $(1, 0)$ tangent to each side of the line $y = x - 1$.

- (12) Consider the nonlinear system

$$\begin{aligned} \frac{dx}{dt} &= x(3 - 3x + 2y), \\ \frac{dy}{dt} &= y(6 - x - y). \end{aligned}$$

- (a) Find all of its equilibrium (critical, stationary) points.

Solution. Stationary points satisfy

$$0 = x(3 - 3x + 2y), \quad 0 = y(6 - x - y).$$

The first equation implies either $x = 0$ or $3 - 3x + 2y = 0$, while the second equation implies either $y = 0$ or $6 - x - y = 0$. If $x = 0$ and $y = 0$ then $(0, 0)$ is a stationary point. If $x = 0$ and $6 - x - y = 0$ then $(0, 6)$ is a stationary point. If $3 - 3x + 2y = 0$ and $y = 0$ then $(1, 0)$ is a stationary point. If $3 - 3x + 2y = 0$ and $6 - x - y = 0$ then upon solving these equations one finds that $(3, 3)$ is a stationary point. All the stationary points of the system are therefore

$$(0, 0), \quad (0, 6), \quad (1, 0), \quad (3, 3).$$

- (b) Compute the coefficient matrix of the linearization (the derivative matrix) at each equilibrium (critical, stationary) point.

Solution. Because

$$\begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix} = \begin{pmatrix} 3x - 3x^2 + 2xy \\ 6y - xy - y^2 \end{pmatrix},$$

the matrix of partial derivatives is

$$\begin{pmatrix} \partial_x f(x, y) & \partial_y f(x, y) \\ \partial_x g(x, y) & \partial_y g(x, y) \end{pmatrix} = \begin{pmatrix} 3 - 6x + 2y & 2x \\ -y & 6 - x - 2y \end{pmatrix}.$$

Evaluating this matrix at each stationary point yields the coefficient matrices

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix} && \text{at } (0, 0), && \mathbf{A} &= \begin{pmatrix} 15 & 0 \\ -6 & -6 \end{pmatrix} && \text{at } (0, 6), \\ \mathbf{A} &= \begin{pmatrix} -3 & 2 \\ 0 & 5 \end{pmatrix} && \text{at } (1, 0), && \mathbf{A} &= \begin{pmatrix} -9 & 6 \\ -3 & -3 \end{pmatrix} && \text{at } (3, 3). \end{aligned}$$

- (c) Identify the type and stability of each equilibrium (critical, stationary) point.

Solution. The coefficient matrix \mathbf{A} at $(0, 0)$ is diagonal, so you can read-off its eigenvalues as 3 and 6. The stationary point $(0, 0)$ is thereby a *nodal source*, which is *unstable*. This is one of the generic types, so it describes the phase-plane portrait of the nonlinear system near $(0, 0)$.

The coefficient matrix \mathbf{A} at $(0, 6)$ is triangular, so you can read-off its eigenvalues as -6 and 15 . The stationary point $(0, 6)$ is thereby a *saddle*, which is *unstable*. This is one of the generic types, so it describes the phase-plane portrait of the nonlinear system near $(0, 6)$.

The coefficient matrix \mathbf{A} at $(1, 0)$ is triangular, so you can read-off its eigenvalues as -3 and 5 . The stationary point $(1, 0)$ is thereby a *saddle*, which is *unstable*. This is one of the generic types, so it describes the phase-plane portrait of the nonlinear system near $(1, 0)$.

The coefficient matrix \mathbf{A} at $(3, 3)$ has eigenvalues that satisfy

$$0 = \det(z\mathbf{I} - \mathbf{A}) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 12z + 45 = (z + 6)^2 + 3^2.$$

Its eigenvalues are thereby $-6 \pm i3$. Because $a_{21} = -3 < 0$, the stationary point $(3, 3)$ is therefore a *clockwise spiral sink*, which is *asymptotically stable*. This is one of the generic types, so it describes the phase-plane portrait of the nonlinear system near $(3, 3)$.

- (d) Sketch a plausible global phase-plane portrait. (Carefully mark the exact lines, curves, and arrows!)

Solution. First observe that the lines $x = 0$ and $y = 0$ are invariant. A trajectory that starts on one of these lines must stay on that line. Along the line $x = 0$ the system reduces to

$$\frac{dy}{dt} = y(6 - y).$$

Along the line $y = 0$ the system reduces to

$$\frac{dx}{dt} = 3x(1 - x).$$

The arrows along these invariant lines can be determined from a phase-line portrait of these reduced systems.

The stationary point $(0, 0)$ is a *nodal source*. The coefficient matrix \mathbf{A} has eigenvalues 3 and 6. Because

$$\mathbf{A} - 3\mathbf{I} = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}, \quad \mathbf{A} - 6\mathbf{I} = \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix},$$

it has the eigenpairs

$$\left(3, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right), \quad \left(6, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

Near there is one trajectory that emerges from $(0, 0)$ along each side of the invariant lines $y = 0$ and $x = 0$. Every other trajectory emerges from $(0, 0)$ tangent to the line $y = 0$, which is the line corresponding to the eigenvalue with the smaller absolute value.

The stationary point $(0, 6)$ is a *saddle*. The coefficient matrix \mathbf{A} has eigenvalues -6 and 15 . Because

$$\mathbf{A} + 6\mathbf{I} = \begin{pmatrix} 21 & 0 \\ -6 & 0 \end{pmatrix}, \quad \mathbf{A} - 15\mathbf{I} = \begin{pmatrix} 0 & 0 \\ -6 & -21 \end{pmatrix},$$

it has the eigenpairs

$$\left(-6, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right), \quad \left(15, \begin{pmatrix} 7 \\ -2 \end{pmatrix} \right)$$

Near $(0, 6)$ there is one trajectory that approaches $(0, 6)$ along each side of the invariant line $x = 0$. There is also one trajectory that emerges from $(0, 6)$ tangent to each side of the line $y = 6 - \frac{2}{7}x$.

The stationary point $(1, 0)$ is a *saddle*. The coefficient matrix \mathbf{A} has eigenvalues -3 and 5 . Because

$$\mathbf{A} + 3\mathbf{I} = \begin{pmatrix} 0 & 2 \\ 0 & 8 \end{pmatrix}, \quad \mathbf{A} - 5\mathbf{I} = \begin{pmatrix} -8 & 2 \\ 0 & 0 \end{pmatrix},$$

it has the eigenpairs

$$\left(-3, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right), \quad \left(5, \begin{pmatrix} 1 \\ 4 \end{pmatrix} \right)$$

Near $(1, 0)$ there is one trajectory that emerges from $(1, 0)$ along each side of the invariant line $y = 0$. There is also one trajectory that approaches $(1, 0)$ tangent to each side of the line $y = 4(x - 1)$.

Finally, the stationary point $(3, 3)$ is a *clockwise spiral sink*. All trajectories in the positive quadrant will spiral into it. A global portrait was sketched during the review session.