# HIGHER-ORDER LINEAR
# ORDINARY DIFFERENTIAL EQUATIONS

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The following is a review of some of the material that we covered on higher-order linear ordinary differential equations. As the presentation of this material in class was different from that in the book, I felt that a written review that closely follows the class presentation might be appreciated.

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## 1. Introdution

**1.1: Normal Form and Solutions.** An $n^{th}$-order linear ordinary differential equation can be brought into the linear normal form

$$\frac{d^n y}{dt^n} + a_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_{n-1}(t)\frac{dy}{dt} + a_n(t)y = f(t)\,. \tag{1.1}$$

Here $a_1(t), \cdots, a_n(t)$ are call *coefficients* while $f(t)$ is called the *forcing* or *driving*. When $f(t) = 0$ the equation is said to be *homogeneous*; otherwise it is said to be *nonhomogeneous*.

> **Definition:** We say that $y = Y(t)$ is a *solution* of (1.1) over an interval $(t_L, t_R)$ provided that:
>
> - the function $Y$ is $n$-times differentiable over $(t_L, t_R)$,
> - the coefficients $a_1(t)$, $a_2(t)$, $\cdots$, $a_n(t)$, and the forcing $f(t)$ are defined over $(t_L, t_R)$,
> - the equation
>
> $$Y^{(n)}(t) + a_1(t)Y^{(n-1)}(t) + \cdots + a_{n-1}(t)Y'(t) + a_n(t)Y(t) = f(t)$$
>
> is satisfied for every $t$ in $(t_L, t_R)$.

The first two bullets simply say that every term appearing in the equation is defined over the interval $(t_L, t_R)$, while the third says the equation is satisfied at each time $t$ in $(t_L, t_R)$.

**1.2: Initial-Value Problem.** An *initial-value problem* associated with (1.1) seeks a solution $y = Y(t)$ of (1.1) that also satisfies the *initial conditions*

$$Y(t_I) = y_0\,, \qquad Y'(t_I) = y_1\,, \qquad \cdots \qquad Y^{(n-1)}(t_I) = y_{n-1}\,, \tag{1.2}$$

for some *initial time* (or *initial point*) $t_I$ and *initial data* (or *initial values*) $y_0, y_1, \cdots, y_{n-1}$. You should know the following basic existence and uniqueness theorem about initial-value problems, which we state without proof.

> **Theorem 1.1 (Basic Existence and Uniqueness Theorem):** Let the functions $a_1, a_2, \cdots, a_n$, and $f$ all be continuous over an interval $(t_L, t_R)$. Then given any initial time $t_I \in (t_L, t_R)$ and any initial data $y_0, y_1, \cdots, y_{n-1}$ there exists a unique solution $y = Y(t)$ of (1.1) that satisfies the initial conditions (1.2). Moreover, this solution has at least $n$ continuous derivatives over $(t_L, t_R)$. If the functions $a_1, a_2, \cdots, a_n$, and $f$ all have $k$ continuous derivatives over $(t_L, t_R)$ then this solution has at least $k + n$ continuous derivatives over $(t_L, t_R)$.

**Remark:** For first-order linear equations ($n = 1$) this theorem was essentially proved when we showed that the unique solution of the initial-value problem

$$\frac{dy}{dt} + a(t)y = f(t)\,, \qquad Y(t_I) = y_0\,,$$

is given by the formula

$$Y(t) = \exp\Big( - \int_{t_I}^{t} a(s)\, ds \Big) \Big[ y_0 + \int_{t_I}^{t} \exp\Big( - \int_{t_I}^{s} a(s_1)\, ds_1 \Big) f(s)\, ds \Big] . \tag{1.3}$$

Because there is no such general formula for the solution of the initial-value problem when $n \geq 2$, the proof of this theorem for higher order equations requires methods beyond the scope of this course.

**Remark:** Later in this chapter we will see that for special choices of coefficients one can construct explicit formulas for the solution of the initial-value problem when $n \geq 2$. Even in such cases we will appeal to this theorem to assert the uniqueness of the solution.

**Remark:** This theorem states the “counting” fact that solutions of $n^{th}$-order linear equations exist and are uniquely specified by $n$ additional pieces of information — specifically, the values of the solution $Y$ and its first $n-1$ derivatives at an initial time $t_I$. It is natural to ask whether one has a similar result if one replaces the $n$ initial conditions (1.2) with any $n$ conditions on $Y$. For example, can one use $n$ conditions that specify the values of $Y$ and some of its derivatives at more than one time? Such a problem is a so-called boundary-value problem. In general solutions to such problems either may not exist or may not be unique. In this course we shall therefore focus on initial-value problems, which are simpler. Boundary-value problems are very important and are studied in more advanced courses.

You should be able to use the Basic Existence and Uniqueness Theorem to argue that certain functions cannot be the solution of a given order of homogeneous linear ordinary differential equations. This is usually argued by contradition.

**Example:** $\sin(t^3)$ cannot be the solution of any equation of the form

$$\frac{d^3 z}{dt^3} + a_1(t) \frac{d^2 z}{dt^2} + a_2(t) \frac{dz}{dt} + a_3(t) z = 0 \,,$$

where $a_1$, $a_2$, and $a_3$, are continuous over an open interval containing 0. Suppose otherwise — namely, suppose that $Z(t) = \sin(t^3)$ satisfies such an equation. Because

$$Z'(t) = 3t^2 \cos(t^3) \,, \qquad Z''(t) = 6t \cos(t^3) - 9t^4 \sin(t^3) \,.$$

we see that $Z(t)$ satisfies the equation and the initial conditions

$$Z(0) = Z'(0) = Z''(0) = 0 \,.$$

However the Basic Existence and Uniqueness Theorem implies that $Z(t) = 0$ is the only solution of the equation that satisfies these initial conditions, which contradicts the fact that $Z(t) = \sin(t^3)$.

**1.3: Intervals of Existence.** You should also be able to use the Basic Existence and Uniqueness Theorem to identify the interval of existence for solutions of (1.1). This is done very much like the way you identified intervals of existence for solutions of first-order linear equations. Specifically, if $Y(t)$ is the solution of the initial value problem (1.1-1.2) then its interval of existence will be $(t_L, t_R)$ whenever:

- all the coefficients and the forcing are continuous over $(t_L, t_R)$,
- the initial time $t_I$ is in $(t_L, t_R)$,
- either a coefficient or the forcing is not defined at each of $t = t_L$ and $t = t_R$.

This is because the first two bullets along with the Basic Existence and Uniqueness Theorem imply that the interval of existence will be at least $(t_L, t_R)$, while the last two bullets along with our definition of solution imply that the interval of existence can be no bigger than $(t_L, t_R)$. This argument works when $t_L = -\infty$ or $t_R = \infty$.

**Example:** Consider the initial value problem

$$\frac{d^3x}{dt^3} + \frac{1}{t^2-4}x = \cos(t)\,, \qquad x(1) = 3\,, \quad x'(1) = 0\,, \quad x''(1) = 0\,.$$

The coefficient and forcing are continuous over $(-2, 2)$; the initial time is $t = 1$, which is in $(-2, 2)$; and the coefficient is not defined at $t = -2$ and at $t = 2$. The interval of existence of the solution is therefore $(-2, 2)$.

**Example:** Consider the initial value problem

$$\frac{d^4y}{dt^4} + \frac{1}{t-4}\frac{dy}{dt} = \frac{e^t}{2+t}\,, \qquad y(0) = y'(0) = y''(0) = y'''(0) = 0\,.$$

The coefficient and forcing are continuous over $(-2, 4)$; the initial time is $t = 0$, which is in $(-2, 4)$; the coefficient is not defined at $t = 4$ while the forcing is not defined at $t = -2$. The interval of existence of the solution is therefore $(-2, 4)$.

**Example:** Consider the initial value problem

$$\frac{d^4y}{dt^4} + \frac{1}{t-4}\frac{dy}{dt} = \frac{e^t}{2+t}\,, \qquad y(6) = y'(6) = y''(6) = y'''(6) = 0\,.$$

The coefficient and forcing are continuous over $(4, \infty)$; the initial time is $t = 6$, which is in $(4, \infty)$; the coefficient is not defined at $t = 4$. The interval of existence of the solution is therefore $(4, \infty)$.

## 2. Homogeneous Equations: General Theory

**2.1: Linear Superposition.** Before we examine the general case, we first study homogeneous linear equations. These have the normal form

$$\frac{d^n y}{dt^n} + a_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_{n-1}(t)\frac{dy}{dt} + a_n(t)y = 0\,. \tag{2.1}$$

We will assume throughout this section that the coefficients $a_1, a_2, \cdots, a_n$ are continuous over an interval $(t_L, t_R)$, so that Therorem 1.1 can be applied. We will exploit the following property of homogeneous equations.

> **Theorem 2.1 (Linear Superposition):** If $Y_1(t)$ and $Y_2(t)$ are solutions of (2.1) then so is
>
> $$c_1 Y_1(t) + c_2 Y_2(t)\,,$$
>
> for any values of the constants $c_1$ and $c_2$. More generally, if $Y_1(t)$, $Y_2(t)$, $\cdots$, $Y_m(t)$ are solutions of (2.1) then so is
>
> $$c_1 Y_1(t) + c_2 Y_2(t) + \cdots + c_m Y_m(t)\,,$$
>
> for any values of the constants $c_1$, $c_2$, $\cdots$, $c_m$.

**Remark:** This theorem states that any linear combination of solutions of (2.1) is also a solution of (2.1). It thereby provides a way to construct a whole family of solutions from a finite number of them.

Suppose you know $n$ "different" solutions of (2.1), $Y_1(t)$, $Y_2(t)$, $\cdots$, $Y_n(t)$. It is natural to ask if you can construct the solution of the initial-value problem as a linear combination of $Y_1(t)$, $Y_2(t)$, $\cdots$, $Y_n(t)$. Set

$$Y(t) = c_1 Y_1(t) + c_2 Y_2(t) + \cdots + c_n Y_n(t)\,.$$

By the superposition theorem this is a solution of (2.1). One only has to check that values of $c_1$, $c_2$, $\cdot$, $c_n$ can be found such that $Y(t)$ satisfies the initial conditions

$$\begin{aligned}
y_0 &= Y(t_I) = c_1 Y_1(t_I) + c_2 Y_2(t_I) + \cdots + c_n Y_n(t_I)\,,\\
y_1 &= Y'(t_I) = c_1 Y_1'(t_I) + c_2 Y_2'(t_I) + \cdots + c_n Y_n'(t_I)\,,\\
&\ \ \vdots \qquad\quad \vdots\\
y_{n-1} &= Y^{(n-1)}(t_I) = c_1 Y_1^{(n-1)}(t_I) + c_2 Y_2^{(n-1)}(t_I) + \cdots + c_n Y_n^{(n-1)}(t_I)\,.
\end{aligned} \tag{2.2}$$

This is a system of $n$ linear algebraic equations for the $n$ unknowns $c_1$, $c_2$, $\cdot$, $c_n$. It seems likely that one can often solve this system.

**Example:** One can check that $e^{2t}$ and $e^{-t}$ are solutions of

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0\,.$$

Let's find $c_1$ and $c_2$ such that $Y(t) = c_1 e^{2t} + c_2 e^{-t}$ satifies the initial conditions

$$Y(0) = y_0\,, \qquad Y'(0) = y_1\,.$$

Because $Y'(t) = c_1 2e^{2t} - c_2 e^{-t}$, these initial condition become

$$y_0 = c_1 + c_2\,, \qquad y_1 = 2c_1 - c_2\,.$$

These can be solved to find

$$c_1 = \frac{y_0 + y_1}{3}\,, \qquad c_2 = \frac{2y_0 - y_1}{3}\,.$$

Hence, for any choice of $y_0$ and $y_1$ the solution of the initial value problem is given by

$$Y(t) = \frac{y_0 + y_1}{3}\, e^{2t} + \frac{2y_0 - y_1}{3}\, e^{-t}\,.$$

**Example:** One can check that $\cos(2t)$ and $\sin(2t)$ are solutions of

$$\frac{d^2y}{dt^2} + 4y = 0\,.$$

Let's find $c_1$ and $c_2$ such that $Y(t) = c_1 \cos(2t) + c_2 \sin(2t)$ satifies the initial conditions

$$Y(0) = y_0\,, \qquad Y'(0) = y_1\,.$$

Because $Y'(t) = -2c_1 \sin(2t) + 2c_2 \cos(2t)$, these initial condition become

$$y_0 = c_1\,, \qquad y_1 = 2c_2\,.$$

These can be easily solved to find

$$c_1 = y_0\,, \qquad c_2 = \frac{y_1}{2}\,.$$

Hence, for any choice of $y_0$ and $y_1$ the solution of the initial value problem is given by

$$Y(t) = y_0 \cos(2t) + y_1 \frac{\sin(2t)}{2}\,.$$

**Example:** One can check that $t$ and $t^2 - 1$ are solutions of

$$(1+t^2)\frac{d^2y}{dt^2} - 2t\,\frac{dy}{dt} + 2y = 0\,.$$

Let's find $c_1$ and $c_2$ such that $Y(t) = c_1 t + c_2(t^2 - 1)$ satifies the initial conditions

$$Y(1) = y_0\,, \qquad Y'(1) = y_1\,.$$

Because $Y'(t) = c_1 + 2c_2 t$, these initial condition become

$$y_0 = c_1\,, \qquad y_1 = c_1 + 2c_2\,.$$

These can be solved to find

$$c_1 = y_0\,, \qquad c_2 = \frac{y_1 - y_0}{2}\,.$$

Hence, for any choice of $y_0$ and $y_1$ the solution of the initial value problem is given by

$$Y(t) = y_0 t + \frac{y_1 - y_0}{2}(t^2 - 1)\,.$$

**Example:** One can check that $e^{4t}$, $e^{3t}$, and $e^{-t}$ are solutions of

$$\frac{d^3y}{dt^3} - 6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 12y = 0\,.$$

Let's find $c_1$, $c_2$, and $c_3$ such that $Y(t) = c_1 e^{4t} + c_2 e^{3t} + c_3 e^{-t}$ satifies the initial conditions

$$Y(0) = y_0\,, \qquad Y'(0) = y_1\,, \qquad Y''(0) = y_2\,.$$

Because

$$\begin{aligned} Y'(t) &= c_1 4e^{4t} + c_2 3e^{3t} - c_3 e^{-t}\,,\\ Y''(t) &= c_1 16e^{4t} + c_2 9e^{3t} + c_3 e^{-t}\,, \end{aligned}$$

these initial condition become

$$\begin{aligned} y_0 &= c_1 + c_2 + c_3\,,\\ y_1 &= 4c_1 + 3c_2 - c_3\,,\\ y_1 &= 16c_1 + 9c_2 + c_3\,. \end{aligned}$$

These can be solved to find

$$c_1 = \frac{-3y_0 - 2y_1 + y_2}{5}\,, \qquad c_2 = \frac{4y_0 + 3y_1 - y_2}{4}\,, \qquad c_3 = \frac{12y_0 - 7y_1 + y_2}{20}\,.$$

Hence, for any choice of $y_0$, $y_1$, and $y_2$ the solution of the initial value problem is given by

$$Y(t) = \frac{-3y_0 - 2y_1 + y_2}{5}\,e^{4t} + \frac{4y_0 + 3y_1 - y_2}{4}\,e^{3t} + \frac{12y_0 - 7y_1 + y_2}{20}\,e^{-t}\,.$$

**2.2: Wronskians.** System (2.2) will have a unique solution for every set of initial data $y_0, y_1, \cdots, y_{n-1}$ if and only if

$$\det\begin{pmatrix} Y_1(t_I) & Y_2(t_I) & \cdots & Y_n(t_I) \\ Y_1'(t_I) & Y_2'(t_I) & \cdots & Y_n'(t_I) \\ \vdots & \vdots & \ddots & \vdots \\ Y_1^{(n-1)}(t_I) & Y_2^{(n-1)}(t_I) & \cdots & Y_n^{(n-1)}(t_I) \end{pmatrix} \neq 0\,. \tag{2.3}$$

This follows from Theorem A.1 which is given in Appendix A on linear algebraic systems. In this section we explore this condition further. We begin with a definition.

**Definition:** Given any $n$ functions $Y_1, Y_2, \cdots, Y_n$ that are $n-1$ times differentiable over an interval $(t_L, t_R)$, a new function $W[Y_1, Y_2, \cdots, Y_n]$, called the *Wronskian* of $Y_1, Y_2, \cdots, Y_n$, is defined over $(t_L, t_R)$ by

$$W[Y_1, Y_2, \cdots, Y_n](t) = \det\begin{pmatrix} Y_1(t) & Y_2(t) & \cdots & Y_n(t) \\ Y_1'(t) & Y_2'(t) & \cdots & Y_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ Y_1^{(n-1)}(t) & Y_2^{(n-1)}(t) & \cdots & Y_n^{(n-1)}(t) \end{pmatrix}\,. \tag{2.4}$$

Condition (2.3) can then be recast as simply

$$W[Y_1, Y_2, \cdots, Y_n](t_I) \neq 0\,. \tag{2.5}$$

It is natural to ask whether condition (2.5) can hold for some initial time $t_I$ but not for other times. The following result will allow us to show that this is not the case.

**Theorem 2.2 (Abel's Theorem):** If $Y_1, Y_2, \cdots, Y_n$ are solutions of (2.1) then $W[Y_1, Y_2, \cdots, Y_n]$ satisfies the first-order linear equation

$$\frac{d}{dt} W[Y_1, Y_2, \cdots, Y_n] + a_1(t) W[Y_1, Y_2, \cdots, Y_n] = 0\,, \tag{2.6}$$

whereby formula (1.3) implies that

$$W[Y_1, Y_2, \cdots, Y_n](t) = W[Y_1, Y_2, \cdots, Y_n](t_I)\, \exp\!\left( -\int_{t_I}^{t} a_1(s)\, ds \right). \tag{2.7}$$

**Proof for Second Order Case:** We will not give a proof of Abel's Theorem in its general setting because to do so would require more properties of determinants than we will cover in this course. We will however give a proof for the second order case, which is the case that you will encounter most often in this course.

Let $Y_1$ and $Y_2$ be two solutions of the second order homogeneous linear equation

$$\frac{d^2y}{dt^2} + a_1(t)\frac{dy}{dt} + a_2(t)y = 0\,.$$

Their Wronskian is given by

$$W[Y_1, Y_2](t) = \det\begin{pmatrix} Y_1(t) & Y_2(t) \\ Y_1'(t) & Y_2'(t) \end{pmatrix} = Y_1(t)Y_2'(t) - Y_1'(t)Y_2(t)\,.$$

Differentiating this formula and then using the differential equation to eliminate $Y_1''(t)$ and $Y_2''(t)$ yields

$$\begin{aligned}
\frac{d}{dt}W[Y_1, Y_2](t) &= Y_1'(t)Y_2'(t) + Y_1(t)Y_2''(t) - Y_1'(t)Y_2'(t) - Y_1''(t)Y_2(t) \\
&= Y_1(t)Y_2''(t) - Y_1''(t)Y_2(t) \\
&= Y_1(t)\big(-a_1(t)Y_2'(t) - a_2(t)Y_2(t)\big) - \big(-a_1(t)Y_1'(t) - a_2(t)Y_1(t)\big)Y_2(t) \\
&= -a_1(t)\big(Y_1(t)Y_2'(t) - Y_1'(t)Y_2(t)\big) - a_2(t)\big(Y_1(t)Y_2(t) - Y_1(t)Y_2(t)\big) \\
&= -a_1(t)W[Y_1, Y_2](t)\,,
\end{aligned}$$

which is equivalent to the first-order equation (2.6) asserted in Abel's Theorem. □

**Exercise:** Give a proof of Abel's Theorem for the third order case along the lines of the one above for the second order case.

An important consequence of Abel's Theorem is that the Wronskian of $n$ solutions of (2.1) is either always zero or never zero.

> **Theorem 2.3:** If $Y_1, Y_2, \cdots, Y_n$ are solutions of (2.1) over an interval $(t_L, t_R)$ then their Wronskian $W[Y_1, Y_2, \cdots, Y_n]$ is either zero everywhere in $(t_L, t_R)$ or zero nowhere in $(t_L, t_R)$.

**Proof:** Suppose that $W[Y_1, Y_2, \cdots, Y_n](t_I) = 0$ for some $t_I$ in $(t_L, t_R)$. Then formula (2.7) immediately implies that $W[Y_1, Y_2, \cdots, Y_n](t) = 0$ everywhere in $(t_L, t_R)$. On the other hand, suppose that $W[Y_1, Y_2, \cdots, Y_n](t_I) \neq 0$ for some $t_I$ in $(t_L, t_R)$. Then because the exponential factor in formula (2.7) is always positive, the formula implies that $W[Y_1, Y_2, \cdots, Y_n](t) \neq 0$ everywhere in $(t_L, t_R)$. □

**2.3: Fundamental Sets of Solutions and General Solutions.** Theorem 2.3 shows us that either condition (2.5) holds everywhere $Y_1, Y_2, \cdots, Y_n$ are defined, or it holds nowhere. This means that when the Wronskian $W[Y_1, Y_2, \cdots, Y_n]$ is nonzero you can always find the unique solution of any initial value problem for any initial time $t_I$ and any initial data $y_0$, $y_1$, $\cdots$, $y_n$. This fact motivates the following definition.

> **Definition:** A set of $n$ solutions of an $n^{th}$ order homogeneous linear ordinary differential equation is said to be *fundamental* if its Wronskian is nonzero.

The importance of this concept is evident in the following.

> **Theorem 2.4:** Let $Y_1$, $Y_2$, $\cdots$, $Y_n$ be a fundamental set of solutions of equation (2.1) over the interval $(t_L, t_R)$. Then every solution of (2.1) over the interval $(t_L, t_R)$ can be expressed as a unique linear combination of $Y_1$, $Y_2$, $\cdots$, $Y_n$.

**Proof:** Let $Y(t)$ be any solution of (2.1) over $(t_L, t_R)$. Consider the $n$-parameter family

$$c_1 Y_1(t) + c_2 Y_2(t) + \cdots + c_n Y_n(t)\,. \tag{2.8}$$

Because $Y_1, Y_2, \cdots, Y_n$ is a fundamental set of solutions, we know $W[Y_1, Y_2, \cdots, Y_n](t_I) \neq 0$ for any time $t_I$ in $(t_L, t_R)$. There is therefore a unique set of values for $c_1$, $c_2$, $\cdots$, $c_n$ such that (2.8) will match the initial values $Y(t_I)$, $Y'(t_I)$, $\cdots$, $Y^{(n-1)}(t_I)$. Because both $Y(t)$ and (2.8) for these values of $c_1$, $c_2$, $\cdots$, $c_n$ are solutions of (2.1) and they satisfy the same initial values at $t_I$, the uniqueness assertion of Theorem 1.1 implies they are equal. □

Theorem 2.4 motivates the following definition.

> **Definition:** If $Y_1$, $Y_2$, $\cdots$, $Y_n$ is a fundamental set of solutions of an $n^{th}$ order homogeneous linear ordinary differential equation then the $n$-parameter family (2.8) is called a *general solution* of the equation.

**Example:** You can check that $Y_1(t) = e^{2t}$ and $Y_2(t) = e^{-t}$ are solutions of

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0\,.$$

They are a fundamental set of solutions because

$$\begin{aligned} W[Y_1, Y_2](t) &= \det\begin{pmatrix} Y_1(t) & Y_2(t) \\ Y_1'(t) & Y_2'(t) \end{pmatrix} = \det\begin{pmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{pmatrix} \\ &= -e^{2t}e^{-t} - 2e^{2t}e^{-t} = -3e^{t} \neq 0\,. \end{aligned}$$

A general solution is therefore $c_1 e^{2t} + c_2 e^{-t}$.

**Example:** You can check that $Y_1(t) = \cos(2t)$ and $Y_2(t) = \sin(2t)$ are solutions of

$$\frac{d^2y}{dt^2} + 4y = 0\,.$$

They are a fundamental set of solutions because

$$\begin{aligned} W[Y_1, Y_2](t) &= \det\begin{pmatrix} Y_1(t) & Y_2(t) \\ Y_1'(t) & Y_2'(t) \end{pmatrix} = \det\begin{pmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{pmatrix} \\ &= 2\cos(2t)^2 + 2\sin(2t)^2 = 2 \neq 0\,. \end{aligned}$$

A general solution is therefore $c_1 \cos(2t) + c_2 \sin(2t)$.

**Example:** You can check that $Y_1(t) = t$ and $Y_2(t) = t^2 - 1$ are solutions of

$$(1 + t^2)\frac{d^2y}{dt^2} - 2t\,\frac{dy}{dt} + 2y = 0\,.$$

They are a fundamental set of solutions because

$$W[Y_1, Y_2](t) = \det\begin{pmatrix} Y_1(t) & Y_2(t) \\ Y_1'(t) & Y_2'(t) \end{pmatrix} = \det\begin{pmatrix} t & t^2 - 1 \\ 1 & 2t \end{pmatrix} = 2t^2 - (t^2 - 1) = t^2 + 1 \neq 0\,.$$

A general solution is therefore $c_1 t + c_2(t^2 - 1)$.

**Example:** You can check that $Y_1(t) = e^{4t}$, $Y_2(t) = e^{3t}$, and $Y_3(t) = e^{-t}$ are solutions of

$$\frac{d^3y}{dt^3} - 6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 12y = 0\,.$$

They are a fundamental set of solutions because

$$\begin{aligned} W[Y_1, Y_2, Y_3](t) &= \det\begin{pmatrix} Y_1(t) & Y_2(t) & Y_3(t) \\ Y_1'(t) & Y_2'(t) & Y_3'(t) \\ Y_1''(t) & Y_2''(t) & Y_3''(t) \end{pmatrix} = \det\begin{pmatrix} e^{4t} & e^{3t} & e^{-t} \\ 4e^{4t} & 3e^{3t} & -e^{-t} \\ 16e^{4t} & 9e^{3t} & e^{-t} \end{pmatrix} \\ &= 3e^{4t}e^{3t}e^{-t} - 16e^{4t}e^{3t}e^{-t} + 36e^{4t}e^{3t}e^{-t} \\ &\quad - 48e^{4t}e^{3t}e^{-t} + 9e^{4t}e^{3t}e^{-t} - 4e^{4t}e^{3t}e^{-t} \\ &= (3 - 16 + 36 - 48 + 9 - 4)e^{6t} = -20e^{6t} \neq 0\,. \end{aligned}$$

A general solution is therefore $c_1 e^{4t} + c_2 e^{3t} + c_3 e^{-t}$.

**2.4: Linear Independence of Solutions.** In the last setion we defined fundamental sets of solutions of a homogeneous linear ordinary differential equation in terms of nonzero Wronskians. Here we develop another characterization of fundamental sets of solutions based on the following notions.

> **Definition:** Functions $Y_1$, $Y_2$, $\cdots$, $Y_m$ defined over an interval $(t_L, t_R)$ are said to be *linearly dependent* if there exists constants $c_1$, $c_2$, $\cdots$, $c_m$, not all zero, such that
>
> $$0 = c_1 Y_1(t) + c_2 Y_2(t) + \cdots + c_m Y_m(t) \quad \text{for every } t \text{ in } (t_L, t_R)\,. \tag{2.9}$$
>
> Otherwise they are said to be *linearly independent.*

If $Y_1$, $Y_2$, $\cdots$, $Y_m$ are linearly dependent then for any $c_k$ that is nonzero, one can solve (2.9) for $Y_k(t)$ as a linear combination of the other functions. For example, if $c_1 \neq 0$ then

$$Y_1(t) = -\frac{c_2}{c_1}\,Y_2(t) - \cdots - \frac{c_m}{c_1}\,Y_m(t) \quad \text{for every } t \text{ in } (t_L, t_R)\,.$$

Because there is at least one nonzero $c_k$, this can always be done for some $Y_k$.

**Example:** The functions $\cos(2t)$, $\cos(t)^2$ and 1 are linearly dependent over $(-\infty, \infty)$ because

$$\cos(2t) = \cos(t)^2 - \sin(t)^2 = 2\cos(t)^2 - 1\,.$$

**Remark:** If one of the functions $Y_1$, $Y_2$, $\cdots$, $Y_m$ is identically zero over $(t_L, t_R)$ then the set is linearly dependent. For example, suppose that $Y_1(t) = 0$ for every $t$ in $(t_L, t_R)$. Then (2.9) holds with $c_1 = 1$ and $c_2 = \cdots = c_m = 0$.

**Remark:** Two functions $Y_1$ and $Y_2$, neither of which is identically zero, are linearly dependent if and only if they are proportional to each other.

**Example:** The functions $t$ and $t^2$ are linearly independent over $(0, 1)$ because they are not proportional to each other. If you think graphically then there is clearly no constant $k$ such that $t^2 = kt$ for every $t$ in $(0, 1)$ because the parabola $y = t^2$ is not a line. Hence, these functions are not linearly dependent.

A good way to generally approach establishing linear independence is the following. A set functions $Y_1$, $Y_2$, $\cdots$, $Y_m$ defined over an interval $(t_L, t_R)$ is linearly independent if the linear relation (2.9) can only hold when $c_1 = c_2 = \cdots = c_m = 0$. When a set of functions is linearly independent there are many ways to show this.

**Example:** The functions 1, $t$ and $t^2$ are linearly independent over $(-\infty, \infty)$. We show this by supposing the linear relation

$$0 = c_1 + c_2 t + c_3 t^2 \qquad \text{for every } t \text{ in } (-\infty, \infty)\,.$$

If we set $t = 0$, $t = 1$, and $t = -1$ into this relation, we obtain the linear algebraic system

$$\begin{aligned} 0 &= c_1\,, \\ 0 &= c_1 + c_2 + c_3\,, \\ 0 &= c_1 - c_2 + c_3\,. \end{aligned}$$

This can be easily solved to show that $c_1 = c_2 = c_3 = 0$, whereby you conclude that 1, $t$ and $t^2$ are linearly independent. A similar argument works if you had chosen to evaluate the linear relations at any other three distinct points, say $t = 2$, $t = 4$, and $t = 6$. We chose to use $t = 0$, $t = 1$, and $t = -1$ because they led to a simple linear algebraic system.

An alternative approach to the above example is to differentiate the linear relation twice with respect to $t$, thereby obtaining

$$\begin{aligned} 0 &= c_1 + c_2 t + c_3 t^2\,, \\ 0 &= c_2 + 2c_3 t\,, \qquad\qquad \text{for every } t \text{ in } (-\infty, \infty)\,. \\ 0 &= 2c_3\,, \end{aligned}$$

If we set $t = 0$ into these equations we immediately see that $c_1 = c_2 = c_3 = 0$, whereby we conclude that 1, $t$ and $t^2$ are linearly independent. We can generalize this approach as follows.

**Theorem 2.5:** If $Y_1$, $Y_2$, $\cdots$, $Y_m$ is a set of $m-1$ times differentiable functions over an interval $(t_L, t_R)$ such that $W[Y_1, Y_2, \cdots, Y_m](t_I) \neq 0$ for some $t_I$ in $(t_L, t_R)$ then they are linearly independent.

**Proof:** We show this by supposing the linear relation

$$0 = c_1 Y_1(t) + c_2 Y_2(t) + \cdots + c_m Y_m(t) \quad \text{for every } t \text{ in } (t_L, t_R)\,.$$

If we differentiate this relation $m-1$ times with respect to $t$ and evaluate the resulting relationships at $t = t_I$, we obtain the linear algebraic system

$$\begin{aligned}
0 &= c_1 Y_1(t_I) + c_2 Y_2(t_I) + \cdots + c_m Y_m(t_I)\,,\\
0 &= c_1 Y_1'(t_I) + c_2 Y_2'(t_I) + \cdots + c_m Y_m'(t_I)\,,\\
&\vdots\\
0 &= c_1 Y_1^{(m-1)}(t_I) + c_2 Y_2^{(m-1)}(t_I) + \cdots + c_m Y_m^{(m-1)}(t_I)\,.
\end{aligned}$$

Because $W[Y_1, Y_2, \cdots, Y_m](t_I) \neq 0$, it follows from Theorem A.1 of Appendix A that $c_1 = c_2 = \cdots = c_m = 0$ is the only solution to this system, from which we conclude the functions $Y_1$, $Y_2$, $\cdots$, $Y_m$ are linearly independent. $\square$

It is natural to ask if linear independence implies having a Wronskian that is nonzero somewhere (or what is the same, if having a Wronskian that is zero everywhere implies linear dependence.) The following example shows that this is not the case.

**Example:** Let $Y_1(t) = t^2$ and $Y_2(t) = |t|t$ over $(-\infty, \infty)$. Because $Y_1'(t) = 2t$ and $Y_2'(t) = 2|t|$ over $(-\infty, \infty)$, we have

$$\begin{aligned}
W[Y_1, Y_2](t) &= \det\begin{pmatrix} Y_1(t) & Y_2(t) \\ Y_1'(t) & Y_2'(t) \end{pmatrix} = \det\begin{pmatrix} t^2 & |t|t \\ 2t & 2|t| \end{pmatrix}\\
&= 2|t|t^2 - 2|t|t^2 = 0 \qquad \text{for every } t \text{ in } (-\infty, \infty)\,.
\end{aligned}$$

However, it is clear that $Y_1$ and $Y_2$ are not porportional, and therefore are linearly independent even though their Wronskian is zero everywhere. Alternatively, you could argue they are linearly independent by first supposing the linear relation

$$0 = c_1 t^2 + c_2 |t| t \qquad \text{for every } t \text{ in } (-\infty, \infty)\,.$$

If we set $t = 1$ and $t = -1$ into this relation, we obtain the linear algebraic system

$$0 = c_1 + c_2\,, \qquad 0 = c_1 - c_2\,.$$

This can be easily solved to show that $c_1 = c_2 = 0$, whereby you conclude that $Y_1$ and $Y_2$ are linearly independent.

The above example shows that a set of linearly independent functions can have a Wronskian that is zero everywhere. However, as the following theorem shows, this cannot happen for sets of $n$ solutions of an $n^{th}$ order homogeneous linear ordinary differential equation.

> **Theorem 2.6:** If $Y_1, Y_2, \cdots, Y_n$ are solutions of (2.1) over an interval $(t_L, t_R)$ then the following properties are equivalent:
>
> (i) $W[Y_1, Y_2, \cdots, Y_n]$ is nonzero everywhere in $(t_L, t_R)$,
> (ii) $W[Y_1, Y_2, \cdots, Y_n]$ is nonzero somewhere in $(t_L, t_R)$,
> (iii) $Y_1, Y_2, \cdots, Y_n$ are linearly independent.

**Remark:** This is the same as saying that following properties are equivalent:

(i′) $W[Y_1, Y_2, \cdots, Y_n]$ is zero somewhere in $(t_L, t_R)$,
(ii′) $W[Y_1, Y_2, \cdots, Y_n]$ is zero everywhere in $(t_L, t_R)$,
(iii′) $Y_1, Y_2, \cdots, Y_n$ are linearly dependent.

The above properties are simply the negations of (i), (ii), and (iii) respectively.

**Remark:** This theorem shows that properties (i), (ii), and (iii) are all equivalent to $Y_1, Y_2, \cdots, Y_n$ being a fundamental set of solutions to (2.1). The equivalence of (i) and (ii) was already established by Theorem 2.3. Below we give an alternative proof of this fact.

**Proof:** It is clear that (i) implies (ii). The fact that (ii) implies (iii) is just Theorem 2.5. Neither of these implications requires the hypothesis that $Y_1, Y_2, \cdots, Y_n$ are solutions of (2.1). All that remains to be proved is that (iii) implies (i). We will do this by contradiction.

Suppose that $Y_1, Y_2, \cdots, Y_n$ are linearly independent and $W[Y_1, Y_2, \cdots, Y_n](t_I) = 0$ for some $t_I$ in $(t_L, t_R)$. Because

$$\det\begin{pmatrix} Y_1(t_I) & Y_2(t_I) & \cdots & Y_n(t_I) \\ Y_1'(t_I) & Y_2'(t_I) & \cdots & Y_n'(t_I) \\ \vdots & \vdots & \ddots & \vdots \\ Y_1^{(n-1)}(t_I) & Y_2^{(n-1)}(t_I) & \cdots & Y_n^{(n-1)}(t_I) \end{pmatrix} = W[Y_1, Y_2, \cdots, Y_n](t_I) = 0\,,$$

Theorem A.2 of Appendix A implies that the linear algebraic system

$$\begin{aligned} 0 &= c_1 Y_1(t_I) + c_2 Y_2(t_I) + \cdots + c_n Y_n(t_I)\,, \\ 0 &= c_1 Y_1'(t_I) + c_2 Y_2'(t_I) + \cdots + c_n Y_n'(t_I)\,, \\ &\vdots \\ 0 &= c_1 Y_1^{(n-1)}(t_I) + c_2 Y_2^{(n-1)}(t_I) + \cdots + c_n Y_n^{(n-1)}(t_I)\,, \end{aligned} \tag{2.10}$$

has a nonzero solution $c_1$, $c_2$, $\cdots$, $c_n$. Now define

$$Y(t) = c_1 Y_1(t) + c_2 Y_2(t) + \cdots + c_n Y_n(t)\,. \tag{2.11}$$

Because $Y_1, Y_2, \cdots, Y_n$ are solutions of (2.1), Theorem 2.1 (Superposition) implies that $Y$ is also a solution of (2.1). By (2.10) we see that $Y$ satisfies the initial conditions

$$Y(t_I) = 0\,, \qquad Y'(t_I) = 0\,, \qquad \cdots \qquad Y^{(n-1)}(t_I) = 0\,.$$

The uniqueness assertion of Theorem 1.1 then implies that $Y(t) = 0$ for every $t$ in $(t_L, t_R)$. Hence, by (2.11) we have

$$0 = c_1 Y_1(t) + c_2 Y_2(t) + \cdots + c_n Y_n(t)\,,$$

where the $c_1$, $c_2$, $\cdots$, $c_n$ are not all zero. But this implies that $Y_1, Y_2, \cdots, Y_n$ are linearly dependent, which is a contradiction. We therefore conclude that $W[Y_1, Y_2, \cdots, Y_n]$ is nonzero everywhere in $(t_L, t_R)$, thereby showing that (iii) implies (i). $\square$

## 3. Homogeneous Equations with Constant Coefficients

**3.1: Characteristic Polynomials and the Key Identity.** In Section 2 we saw how to construct general solutions of homogeneous linear differential equations given a fundamental set of solutions. While there is no general recipe for constructing fundamental sets of solutions, it can be done for special cases. Here we will study the most important such special case — namely, the case where the coefficients are constants. In that case (2.1) becomes

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy}{dt} + a_n y = 0 \,, \tag{3.1}$$

where $a_1, a_2, \cdots, a_n$ are constants. We can express this compactly as

$$\mathbf{L} y = 0 \,, \tag{3.2}$$

where $\mathbf{L}$ is the $n^{th}$ order differential operator

$$\mathbf{L} = \frac{d^n}{dt^n} + a_1 \frac{d^{n-1}}{dt^{n-1}} + \cdots + a_{n-1} \frac{d}{dt} + a_n \,. \tag{3.3}$$

We will sometimes write

$$\mathbf{L} = P(\mathbf{D}) \,, \quad \text{where} \quad \mathbf{D} = \frac{d}{dt} \,,$$

and $P(z)$ is the $n^{th}$ degree real polynomial

$$P(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n \,. \tag{3.4}$$

This is the *characteristic polynomial* associated with the $n^{th}$ order differential operator $\mathbf{L}$.

Repeated differentiation of the function $e^{zt}$ yields the identities

$$\frac{d}{dt} e^{zt} = z\, e^{zt} \,, \qquad \frac{d^2}{dt^2} e^{zt} = z^2\, e^{zt} \,, \qquad \cdots \qquad \frac{d^k}{dt^k} e^{zt} = z^k\, e^{zt} \,, \qquad \cdots \,,$$

for every positive integer $k$. Hence, we find that

$$\begin{aligned} P(\mathbf{D}) e^{zt} &= \frac{d^n}{dt^n} e^{zt} + a_1 \frac{d^{n-1}}{dt^{n-1}} e^{zt} + \cdots + a_{n-1} \frac{d}{dt} e^{zt} + a_n e^{zt} \\ &= z^n e^{zt} + a_1 z^{n-1} e^{zt} + \cdots + a_{n-1} z\, e^{zt} + a_n e^{zt} = P(z)\, e^{zt} \,, \end{aligned}$$

We have derived the KEY identity

$$\mathbf{L} e^{zt} = P(\mathbf{D}) e^{zt} = P(z)\, e^{zt} \,. \tag{3.5}$$

In the remainder of this section we will show how to use the KEY identity to find a recipe for the general solution of homogeneous equation $\mathbf{L} y = 0$.

**3.2: Real Roots of Characteristic Polynomials.** If $r$ is a real root of $P(z)$ (i.e. if $P(r) = 0$) then the KEY identity (3.5) implies that

$$\mathbf{L}e^{rt} = P(r)e^{rt} = 0\,,$$

whereby $e^{rt}$ is a solution of the homogeneous equation $\mathbf{L}y = 0$.

It follows from the above observation that if a characteristic polynomial has $n$ simple real roots $r_1,\ r_2,\ \cdots,\ r_n$ then one has $n$ solutions of the homogeneous equation $\mathbf{L}y = 0$. It can be shown that these solutions are independent. For example, when $n = 3$ one sees that the Wronskian is

$$\det\begin{pmatrix} e^{r_1 t} & e^{r_2 t} & e^{r_3 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} & r_3 e^{r_3 t} \\ r_1^2 e^{r_1 t} & r_2^2 e^{r_2 t} & r_3^2 e^{r_3 t} \end{pmatrix} = (r_3 - r_2)(r_2 - r_1)(r_3 - r_1)e^{(r_1 + r_2 + r_3)t} \neq 0\,.$$

The argument for independence when $n \geq 4$ will not be given here. Given this independence however, one thereby concludes that the general solution of $\mathbf{L}y = 0$ is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{r_n t}\,.$$

The most difficult part of applying this recipe is often finding the roots of the characteristic polynomial. Of course, for quadratic polynomials this can be done by completing the square or by using the quadratic formula. In this course characteristic polynomials of degree three or more will generally have some easily found root like 0, ±1, ±2, or ±3. If the coefficients of $P(z)$ are integers, you should first check for roots that are factors of $a_n$. Given that you have found a real root $r$, the characteristic polynomial can be factored as

$$P(z) = (z - r)Q(z)\,,$$

where $Q(z)$ is an $(n-1)^{th}$ degree real polynomial

$$P(z) = z^{n-1} + b_1 z^{n-2} + \cdots + b_{n-2}z + b_{n-1}\,.$$

One thereby reduces the problem of finding roots of $P(z)$ to finding roots of $Q(z)$. If a characteristic polynomial has $n$ simple real roots $r_1,\ r_2,\ \cdots,\ r_n$ then this procedure is repeated until you have completely factored $P(z)$ into the form

$$P(z) = (z - r_1)(z - r_2)\cdots(z - r_n)\,.$$

Of course, if $P(z)$ is given to you in factored form, you can just read off the roots!

**Example:** Find a general solution of

$$\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0\,.$$

The characteristic polynomial is

$$P(z) = z^3 + 2z^2 - z - 2 = (z-1)(z+1)(z+2)\,.$$

Its three roots are 1, $-1$, $-2$. The solution associated with the root 1 is $e^t$. The solution associated with the root $-1$ is $e^{-t}$. The solution associated with the root $-2$ is $e^{-2t}$. The general solution is therefore

$$y = c_1 e^t + c_2 e^{-t} + c_3 e^{-2t}\,.$$

Of course, polynomials of degree $n$ do not generally have $n$ simple real roots. There are two ways this can fail to happen. First, a real root might not be simple — that is, a real root might have multiplicity greater than one. Second, polynomials might have irreducible factors which correspond to complex roots. We first examine how to treat cases with real roots of multiplicity greater than one.

Recall that $r$ is a double real root of $P(z)$ when $(z-r)^2$ is a factor of $P(z)$. This is equivalent to the condition $P(r) = P'(r) = 0$. Differentiation of the KEY identity (3.5) with respect to $z$ gives

$$\mathbf{L}\big(t\,e^{zt}\big) = P(z)\,t\,e^{zt} + P'(z)\,e^{zt}\,.$$

Evaluating this at $z = r$ shows that

$$\mathbf{L}\big(t\,e^{rt}\big) = P(r)\,t\,e^{rt} + P'(r)\,e^{rt} = 0\,.$$

Hence, $e^{rt}$ and $t\,e^{rt}$ are solutions of the homogeneous equation $\mathbf{L}y = 0$. Because

$$\frac{d}{dt}e^{rt} = r\,e^{rt}\,, \qquad \frac{d}{dt}(t\,e^{rt}) = rt\,e^{rt} + e^{rt}\,,$$

the Wronskian of these solutions is

$$\det\begin{pmatrix} e^{rt} & t\,e^{rt} \\ r\,e^{rt} & rt\,e^{rt} + e^{rt} \end{pmatrix} = e^{rt}(rt\,e^{rt} + e^{rt}) - t\,e^{rt} r\,e^{rt} = e^{2rt} \neq 0\,.$$

These solutions are therefore independent.

Recall that $r$ is a triple real root of $P(z)$ when $(z-r)^3$ is a factor of $P(z)$. This is equivalent to the condition $P(r) = P'(r) = P''(r) = 0$. Differentiation of the KEY identity (3.5) twice with respect to $z$ gives

$$\mathbf{L}\big(t^2 e^{zt}\big) = P(z)\,t^2 e^{zt} + 2P'(z)\,t\,e^{zt} + P''(z)\,e^{zt}\,.$$

Evaluating this at $z = r$ shows that

$$\mathbf{L}\big(t^2 e^{rt}\big) = P(r)\,t^2 e^{rt} + 2P'(r)\,t\,e^{rt} + P''(r)\,e^{rt} = 0\,.$$

Hence, $e^{rt}$, $t\,e^{rt}$, and $t^2e^{rt}$ are solutions of the homogeneous equation $\mathbf{L}y = 0$. Because

$$\frac{d}{dt}e^{rt} = r\,e^{rt}\,, \qquad \frac{d}{dt}(t\,e^{rt}) = rt\,e^{rt} + e^{rt}\,, \qquad \frac{d}{dt}(t^2e^{rt}) = rt^2e^{rt} + 2t\,e^{rt}\,,$$

and

$$\frac{d^2}{dt^2}e^{rt} = r^2e^{rt}\,, \quad \frac{d^2}{dt^2}(t\,e^{rt}) = r^2t\,e^{rt} + 2r\,e^{rt}\,, \quad \frac{d^2}{dt^2}(t^2e^{rt}) = r^2t^2e^{rt} + 4rt\,e^{rt} + 2\,e^{rt}\,,$$

the Wronskian of these solutions is

$$\det\begin{pmatrix} e^{rt} & t\,e^{rt} & t^2e^{rt} \\ r\,e^{rt} & rt\,e^{rt} + e^{rt} & rt^2e^{rt} + 2t\,e^{rt} \\ r^2e^{rt} & r^2t\,e^{rt} + 2r\,e^{rt} & r^2t^2e^{rt} + 4rt\,e^{rt} + 2\,e^{rt} \end{pmatrix} = 2e^{3rt} \neq 0\,.$$

These solutions are therefore independent.

More generally, recall that $r$ is a real root of $P(z)$ of multiplicity $m$ when $(z-r)^m$ is a factor of $P(z)$. This is equivalent to the condition $P(r) = P'(r) = \cdots = P^{(m-1)}(r) = 0$. Differentiation of the KEY identity (3.5) $k$ times with respect to $z$ gives

$$\mathbf{L}\big(t^ke^{zt}\big) = P(z)\,t^ke^{zt} + kP'(z)\,t^{k-1}e^{zt} + \cdots + kP^{(k-1)}(z)\,t\,e^{zt} + P^{(k)}(z)\,e^{zt}\,.$$

Evaluating this at $z = r$ when $k = 0, 1, \cdots, m-1$ shows that

$$\mathbf{L}\big(t^ke^{rt}\big) = P(r)\,t^ke^{rt} + kP'(r)\,t^{k-1}e^{rt} + \cdots + kP^{(k-1)}(r)\,t\,e^{rt} + P^{(k)}(r)\,e^{rt} = 0\,.$$

Hence,

$$e^{rt}\,, \qquad t\,e^{rt}\,, \qquad \cdots \qquad t^{m-1}e^{rt}\,,$$

are $m$ solutions of the homogeneous equation $\mathbf{L}y = 0$. These solutions are in fact independent, but we will not show that here.

**Example:** Find a general solution of

$$\mathbf{D}^6y - 5\mathbf{D}^5y + 6\mathbf{D}^4y + 4\mathbf{D}^3y - 8\mathbf{D}^2y = 0\,, \qquad \text{where} \quad \mathbf{D} = \frac{d}{dt}\,.$$

The characteristic polynomial is

$$P(z) = z^6 - 5z^5 + 6z^4 + 4z^3 - 8z^2 = z^2(z+1)(z-2)^3\,.$$

Its six roots (counting multiplicity) are 0, 0, $-1$, 2, 2, 2. The solutions associated with the double root 0 are 1 and $t$. The solution associated with the simple root $-1$ is $e^{-t}$. The solutions associated with the triple root 2 are $e^{2t}$, $t\,e^{2t}$, and $t^2e^{2t}$. The general solution is therefore

$$y = c_1 + c_2t + c_3e^{-t} + c_4e^{2t} + c_5t\,e^{2t} + c_6t^2e^{2t}\,.$$

**3.3: Complex Roots of Characteristic Polynomials.** Consider the problem of finding the general solution of

$$\mathbf{L}y = \mathbf{D}^2 y + 9y = 0\,, \qquad \text{where} \quad \mathbf{D} = \frac{d}{dt}\,.$$

The characteristic polynomial is $P(z) = z^2 + 9$, which clearly has no real roots. However, this can be factored over the complex numbers as

$$P(z) = (z - i3)(z + i3)\,,$$

where $i = \sqrt{-1}$. Its roots are the conjugate pair $i3$ and $-i3$. We claim that $e^{i3t}$ and $e^{-i3t}$ are independent complex-valued solutions of $\mathbf{L}y = 0$. We must first recall what is meant by such complex-valued solutions. We must then see how to generate real-valued solutions from them.

You should recall the Euler identity from your study of calculus. It states that for every real $\theta$ one has

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)\,.$$

This identity is the key to making sense of complex exponentials. In particular, for any real number $s$ one has

$$e^{ist} = \cos(st) + i\sin(st)\,.$$

The derivative of this function with respect to $t$ is then

$$\begin{aligned}\mathbf{D}e^{ist} &= \mathbf{D}\big(\cos(st) + i\sin(st)\big) = \mathbf{D}\cos(st) + i\mathbf{D}\sin(st)\\ &= -s\sin(st) + is\cos(st) = is\big(\cos(st) + i\sin(st)\big) = ise^{ist}\,.\end{aligned}$$

More generally, for any real numbers $r$ and $s$ one has

$$e^{(r+is)t} = e^{rt+ist} = e^{rt}e^{ist} = e^{rt}\big(\cos(st) + i\sin(st)\big)\,.$$

By the product rule and our result that $\mathbf{D}e^{ist} = ise^{ist}$, the derivative of this function with respect to $t$ is

$$\begin{aligned}\mathbf{D}e^{(r+is)t} &= \mathbf{D}\big(e^{rt}e^{ist}\big) = \mathbf{D}\big(e^{rt}\big)e^{ist} + e^{rt}\mathbf{D}\big(e^{ist}\big)\\ &= re^{rt}e^{ist} + ise^{rt}e^{ist} = (r + is)e^{(r+is)t}\,.\end{aligned}$$

We thereby see that for any complex number $z$ we have $\mathbf{D}e^{zt} = ze^{zt}$, from which it follows that for any polynomial $P(z)$ one has

$$P(\mathbf{D})e^{zt} = P(z)e^{zt}\,. \tag{3.6}$$

In particular, the KEY identity holds for any complex $z$!

Let $P(z)$ be the characteristic polynomial of the differential operator $\mathbf{L}$ given by (3.3). Because $P(z)$ has real coefficients, it has the property that

$$P(\overline{z}) = \overline{P(z)} \quad \text{for every complex } z\,,$$

where the bar denotes complex conjugate — i.e. $\overline{X+iY} = X - iY$ for any real $X$ and $Y$. Thus, if $P(r+is) = 0$ then

$$P(r-is) = P(\overline{r+is}) = \overline{P(r+is)} = 0\,.$$

Hence, roots of $P(z)$ come in conjugate pairs; if $r+is$ is a root then so is $r-is$.

By the KEY identity (3.6), if $P(z)$ has a conjugate pair of roots $r+is$ and $r-is$ then $e^{(r+is)t}$ and $e^{(r-is)t}$ are a pair of complex-valued solutions of equation (3.1) — namely, they satisfy

$$\mathbf{L}e^{(r+is)t} = 0\,, \qquad \mathbf{L}e^{(r-is)t} = 0\,. \tag{3.7}$$

Because $e^{(r+is)t} = e^{rt}\cos(st) + ie^{rt}\sin(st)$, its real and imaginary parts are

$$\mathrm{Re}\big(e^{(r+is)t}\big) = e^{rt}\cos(st)\,, \qquad \mathrm{Im}\big(e^{(r+is)t}\big) = e^{rt}\sin(st)\,.$$

Recall that for any complex $Z$ its real and imaginary parts can be expressed as

$$\mathrm{Re}(Z) = \frac{Z+\overline{Z}}{2}\,, \qquad \mathrm{Im}(Z) = \frac{Z-\overline{Z}}{i2}\,.$$

We therefore have

$$\begin{aligned} e^{rt}\cos(st) &= \mathrm{Re}\big(e^{(r+is)t}\big) = \frac{e^{(r+is)t} + e^{(r-is)t}}{2}\,,\\ e^{rt}\sin(st) &= \mathrm{Im}\big(e^{(r+is)t}\big) = \frac{e^{(r+is)t} - e^{(r-is)t}}{i2}\,. \end{aligned}$$

It then follows from (3.7) that

$$\begin{aligned} \mathbf{L}\big(e^{rt}\cos(st)\big) &= \mathbf{L}\left(\frac{e^{(r+is)t} + e^{(r-is)t}}{2}\right) = \frac{1}{2}\Big(\mathbf{L}e^{(r+is)t} + \mathbf{L}e^{(r-is)t}\Big) = 0\,,\\ \mathbf{L}\big(e^{rt}\sin(st)\big) &= \mathbf{L}\left(\frac{e^{(r+is)t} - e^{(r-is)t}}{i2}\right) = \frac{1}{i2}\Big(\mathbf{L}e^{(r+is)t} - \mathbf{L}e^{(r-is)t}\Big) = 0\,. \end{aligned}$$

In other words, when the characteristic polynomial $P(z)$ of the differential operator $\mathbf{L}$ has a conjugate pair of roots $r+is$ and $r-is$ then $e^{rt}\cos(st)$ and $e^{rt}\sin(st)$ are a pair of real-valued solutions of equation (3.1). You can easily check that they are linearly independent when $s \neq 0$.

If $P(z)$ has a conjugate pair of roots $r + is$ and $r - is$ with $s \neq 0$ then it has the pair of complex factors $(z - r - is)$ and $(z - r + is)$. Because

$$(z - r - is)(z - r + is) = (z - r)^2 - (is)^2 = (z - r)^2 + s^2 \,,$$

we see that $P(z)$ has the irreducible real factor $(z - r)^2 + s^2$. Conversely, if $P(z)$ has the irreducible real factor $(z - r)^2 + s^2$ then it has the conjugate pair of roots $r + is$ and $r - is$.

**Example:** Find a general solution of

$$\mathbf{L}y = \mathbf{D}^2 y + 2\mathbf{D}y + 5y = 0 \,, \qquad \text{where} \quad \mathbf{D} = \frac{d}{dt} \,.$$

**Solution:** The characteristic polynomial is

$$P(z) = z^2 + 2z + 5 = (z + 1)^2 + 4 = (z + 1)^2 + 2^2 \,,$$

which has the conjugate pair of roots $-1 + i2$ and $-1 - i2$. A general solution is therefore

$$y = c_1 e^{-t} \cos(2t) + c_2 e^{-t} \sin(2t) \,.$$

**Example:** Find a general solution of

$$\mathbf{L}y = \mathbf{D}^2 y + 9y = 0 \,, \qquad \text{where} \quad \mathbf{D} = \frac{d}{dt} \,.$$

**Solution:** The characteristic polynomial is

$$P(z) = z^2 + 9 = z^2 + 3^2 \,,$$

which has the conjugate pair of roots $i3$ and $-i3$. A general solution is therefore

$$y = c_1 \cos(3t) + c_2 \sin(3t) \,.$$

**Example:** Find a general solution of

$$\mathbf{L}y = (\mathbf{D} + 5)^3 (\mathbf{D}^2 + 4\mathbf{D} + 5)(\mathbf{D}^2 + 4)y = 0 \,, \qquad \text{where} \quad \mathbf{D} = \frac{d}{dt} \,.$$

**Solution:** The characteristic polynomial is

$$P(z) = (z + 5)^3 (z^2 + 4z + 5)(z^2 + 4) = (z + 5)^3 \big((z + 2)^2 + 1^2\big)\big(z^2 + 2^2\big) \,,$$

which has the real roots $-5$, $-5$, $-5$, the conjugate pair of roots $-2 + i$, $-2 - i$, and the conjugate pair of roots $i2$, $-i2$. A general solution is therefore

$$y = c_1 e^{-5t} + c_2 t\, e^{-5t} + c_3 t^2 e^{-5t} + c_4 e^{-2t} \cos(t) + c_5 e^{-2t} \sin(t) + c_6 \cos(2t) + c_7 \sin(2t) \,.$$

The fundamental theorem of algebra says that any polynomial $P(z)$ of degree $n$ can be written as the product of $n$ linear factors:

$$P(z) = (z - z_1)(z - z_2) \cdots (z - z_n)\,,$$

where $z_1$, $z_2$, $\cdots$, $z_n$ are complex numbers that are roots of $P(z)$ — i.e. $P(z_j) = 0$ for each $z_j$. Conversely, if $P(r + is) = 0$ then $r + is = z_j$ for some $z_j$. We say $r + is$ is a root of $P(z)$ of multiplicity $m$ if $r + is = z_j$ for $m$ of the $z_j$. In other words, $r + is$ is a root of $P(z)$ of multiplicity $m$ if $(z - r - is)^m$ is a factor of $P(z)$.

If all the coefficients of a polynomial $P(z)$ are real then it is called a *real polynomial*. Characteristic polynomials of linear differential operators in the form (3.3) are real polynomials. If $P(z)$ is a real polynomial and $r + is$ is a root of $P(z)$ of multiplicity $m$ its conjugate $r - is$ is also a root of $P(z)$ of multiplicity $m$. If $s \neq 0$ then this means that $(z - r - is)^m$ and $(z - r + is)^m$ are distinct complex factors of $P(z)$, which means that $\big((z - r)^2 + s^2\big)^m$ is a real factor of $P(z)$. Conversely, if $\big((z - r)^2 + s^2\big)^m$ is a factor of $P(z)$ for some real $r$ and $s$ and some positive integer $m$ and if $s \neq 0$ then $r + is$ and $r - is$ are distinct roots of $P(z)$ of multiplicity $m$.

**Example:** Find all the roots of $P(z) = (z^3 - 2z^2)(z^2 - 2z + 10)^3(z^2 + 4z + 29)$.

**Solution:** Because the degree of a factored polynomial is the sum of the degrees of its factors, you see that the degree of $P(z)$ is $3 + 6 + 2 = 11$. Because $P(z)$ has degree 11, it must have 11 roots counting multiplicities. Because

$$P(z) = z^2(z - 2)\big((z - 1)^2 + 3^2\big)^3\big((z + 2)^2 + 5^2\big)\,,$$

the 11 roots are 0, 0, 2, $1 \pm i3$, $1 \pm i3$, $1 \pm i3$, $-2 \pm i5$. Here each $r \pm is$ denotes two distinct roots. The real root 0 has multiplicity 2 while the complex roots $1 + i3$ and $1 - i3$ have multiplicity 3.

Now let $P(z)$ be the characteristic polynomial of an $n^{th}$ order linear differential operator $\mathbf{L}$ in the form (3.3). We know that $P(z)$ has $n$ complex roots counting multiplicities. We already know that if $r$ is a real root of $P(z)$ of multiplicity $m$ then $\mathbf{L}y = 0$ has the $m$ linearly independent real solutions given by

$$e^{rt}\,, \qquad t\,e^{rt}\,, \qquad \cdots \qquad t^{m-1}e^{rt}\,. \tag{3.8}$$

Below we will show that if $r \pm is$ is a conjugate pair of roots of $P(z)$ of multiplicity $m$ then $\mathbf{L}y = 0$ has the $2m$ solutions

$$\begin{aligned} &e^{rt}\cos(st)\,, \qquad t\,e^{rt}\cos(st)\,, \qquad \cdots \qquad t^{m-1}e^{rt}\cos(st)\,,\\ &e^{rt}\sin(st)\,, \qquad t\,e^{rt}\sin(st)\,, \qquad \cdots \qquad t^{m-1}e^{rt}\sin(st)\,. \end{aligned} \tag{3.9}$$

The $n$ roots of $P(z)$ therefore generate $n$ solutions of $\mathbf{L}y = 0$ by recipes (3.8) and (3.9). Moreover, it can be shown that these solutions are linearly independent, and thereby are a fundamental set of solutions for the problem.

**Example:** Find a general solution of

$$\mathbf{D}^4 y + 8\mathbf{D}^2 y + 16y = 0\,, \qquad \text{where} \quad \mathbf{D} = \frac{d}{dt}\,.$$

**Solution:** The characteristic polynomial is

$$P(z) = z^4 + 8z^2 + 16 = \big(z^2 + 4\big)^2 = \big(z^2 + 2^2\big)^2\,.$$

Its 4 roots are $\pm i2$, $\pm i2$. A general solution is therefore

$$y = c_1 \cos(2t) + c_2 \sin(2t) + c_3 t\,\cos(2t) + c_4 t\,\sin(2t)\,.$$

**Example:** Find a general solution of

$$\big(\mathbf{D}^3 - 2\mathbf{D}^2\big)\big(\mathbf{D}^2 - 2\mathbf{D} + 10\big)^3\big(\mathbf{D}^2 + 4\mathbf{D} + 29\big)y = 0\,, \qquad \text{where} \quad \mathbf{D} = \frac{d}{dt}\,.$$

**Solution:** The characteristic polynomial is

$$\begin{aligned} P(z) &= (z^3 - 2z^2)(z^2 - 2z + 10)^3(z^2 + 4z + 29) \\ &= z^2(z-2)\big((z-1)^2 + 3^2\big)^3\big((z+2)^2 + 5^2\big)\,. \end{aligned}$$

Its 11 roots are 0, 0, 2, $1 \pm i3$, $1 \pm i3$, $1 \pm i3$, $-2 \pm i5$. A general solution is therefore

$$\begin{aligned} y = c_1 + c_2 t + c_3 e^{2t} + c_4 e^t \cos(3t) + c_5 e^t \sin(3t) + c_6 t\, e^t \cos(3t) + c_7 t\, e^t \sin(3t) \\ + c_8 t^2 e^t \cos(3t) + c_9 t^2 e^t \sin(3t) + c_{10} e^{-2t} \cos(5t) + c_5 e^{-2t} \sin(5t)\,. \end{aligned}$$

Recall that recipe (3.8) was derived by evaluating the KEY identity and its first $m-1$ derivatives at the root $z = r$, and using that fact that $P(r) = P'(r) = \cdots = P^{(m-1)}(r) = 0$. Recipe (3.9) is derived in a similar way. If $r + is$ is a complex root $P(z)$ of multiplicity $m$ then $(z - r - is)^m$ is a factor of $P(z)$. One can differentiate polynomials and $e^{zt}$ with respect to the complex variable $z$ exactly as if it were a real variable. Because $(z-r-is)^m$ is a factor of $P(z)$, you can show that

$$P(r + is) = P'(r + is) = \cdots = P^{m-1}(r + is) = 0\,.$$

Differentiation of the KEY identity (3.6) $k$ times with respect to $z$ gives

$$\mathbf{L}\big(t^k e^{zt}\big) = P(z)\, t^k e^{zt} + kP'(z)\, t^{k-1} e^{zt} + \cdots + kP^{(k-1)}(z)\, t\, e^{zt} + P^{(k)}(z)\, e^{zt}\,.$$

Evaluating this at $z = r + is$ when $k = 0, 1, \cdots, m-1$ shows that

$$\begin{aligned} \mathbf{L}\big(t^k e^{(r+is)t}\big) &= P(r+is)\, t^k e^{(r+is)t} + kP'(r+is)\, t^{k-1} e^{(r+is)t} + \cdots \\ &\qquad \cdots + kP^{(k-1)}(r+is)\, t\, e^{(r+is)t} + P^{(k)}(r+is)\, e^{(r+is)t} = 0\,. \end{aligned}$$

Similarly you can show that $\mathbf{L}\big(t^k e^{(r-is)t}\big) = 0$. Recipe (3.9) then follows by the taking real and imagingary parts of these complex-valued solutions.

## 4. Nonhomogeneous Equations

**4.1: General Theory.** We are now ready to study nonhomogeneous linear equations. These have the normal form

$$\mathbf{L}(t)y = f(t)\,, \tag{4.1}$$

where the differential operator $\mathbf{L}(t)$ has the form

$$\mathbf{L}(t) = \frac{d^n}{dt^n} + a_1(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + a_{n-1}(t)\frac{d}{dt} + a_n(t)\,. \tag{4.2}$$

We will assume throughout this section that the coefficients $a_1, a_2, \cdots, a_n$ and the forcing $f$ are continuous over an interval $(t_L, t_R)$, so that Therorem 1.1 can be applied.

We will exploit the following properties of nonhomogeneous equations.

**Theorem 4.1:** If $Y_1(t)$ and $Y_2(t)$ are solutions of (4.1) then $Z(t) = Y_1(t) - Y_2(t)$ is a solution of the associated homogeneous equation

$$\mathbf{L}(t)Z(t) = 0\,.$$

**Proof:** Because $\mathbf{L}(t)Y_1(t) = f(t)$ and $\mathbf{L}(t)Y_2(t) = f(t)$ one sees that

$$\mathbf{L}(t)Z(t) = \mathbf{L}(t)\big(Y_1(t) - Y_2(t)\big) = \mathbf{L}(t)Y_1(t) - \mathbf{L}(t)Y_2(t) = f(t) - f(t) = 0\,.$$

**Theorem 4.2:** If $Y_P(t)$ is a solution of (4.1) and $Y_H(t)$ is a solution of the associated homogeneous equation $\mathbf{L}(t)Y_H(t) = 0$ then $Y(t) = Y_H(t) + Y_P(t)$ is also a solution of (4.1).

**Proof:** Because $\mathbf{L}(t)Y_H(t) = 0$ and $\mathbf{L}(t)Y_P(t) = f(t)$ one sees that

$$\mathbf{L}(t)Y(t) = \mathbf{L}(t)\big(Y_H(t) + Y_P(t)\big) = \mathbf{L}(t)Y_H(t) + \mathbf{L}(t)Y_P(t) = 0 + f(t) = f(t)\,.$$

Theorem 4.2 suggests that we can construct general solutions of equation (4.1) as follows.

(1) Find the general solution $Y_H(t)$ of the associated homogeneous equation $\mathbf{L}(t)y = 0$.

(2) Find a *particular solution* $Y_P(t)$ of equation (4.1).

(3) Then $Y_H(t) + Y_P(t)$ is a general solution of (4.1).

Of course, step (1) reduces to finding a fundamental set of solutions of the associated homogeneous equation, $Y_1, Y_2, \cdots, Y_n$. Then

$$Y_H(t) = c_1Y_1(t) + c_2Y_2(t) + \cdots + c_nY_n(t)\,.$$

If $\mathbf{L}(t)$ has constant coefficients (so that $\mathbf{L}(t) = \mathbf{L}$) then this can be done by the recipe of Section 3.

**Example.** One can check that $\frac{1}{4}t$ is a particular solution of

$$\frac{d^2y}{dt^2} + 4y = t\,.$$

This equation has constant coefficients. Its characteristic polynomial is $P(z) = z^2 + 4$, which has roots $\pm i2$. A general solution is therefore

$$c_1 \cos(2t) + c_2 \sin(2t) + \tfrac{1}{4}t\,.$$

**Example.** One can check that $-\frac{1}{2}e^t$ is a particular solution of

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = e^t\,.$$

This equation has constant coefficients. Its characteristic polynomial is $P(z) = z^2 - z - 2 = (z-2)(z+1)$, which has roots $-1$ and $2$. A general solution is therefore

$$c_1 e^{-t} + c_2 c^{2t} - \tfrac{1}{2}e^t\,.$$

These examples show that when $\mathbf{L}(t)$ has constant coefficients (so that $\mathbf{L}(t) = \mathbf{L}$), finding $Y_P(t)$ becomes the crux of matter. If $\mathbf{L}(t)$ does not have constant coefficients then a fundamental set of solutions of the associated homogeneous equation will generally be given to you. In that case, finding $Y_P(t)$ again becomes the crux of matter. The remainder of this section will present two methods for finding particular solutions $Y_P(t)$.

**4.2: Undertermined Coefficients.** This method can be used to find particular solutions of

$$\mathbf{L}y = f(t) \tag{4.3}$$

whenever

(1) the differential operator $\mathbf{L}$ has constant coefficients,

$$\mathbf{L} = \frac{d^n}{dt^n} + a_1 \frac{d^{n-1}}{dt^{n-1}} + \cdots + a_{n-1}\frac{d}{dt} + a_n\,, \tag{4.4}$$

(2) the forcing $f(t)$ is a linear combination of functions that have the form

$$t^k e^{rt} \cos(st)\,, \qquad \text{or} \qquad t^k e^{rt} \sin(st)\,, \tag{4.5}$$

where $k$ is a nonnegative integer and $r$ and $s$ are real.

The first of these conditions is always easy to verify. The second might require some use of trigonometric or other identities in order to verify it.

**Example:** The forcing of the equation $\mathbf{L}y = 2e^{2t}$ has the form (4.5) with $k = 0$, $r = 2$ and $s = 0$.

**Example:** The forcing of the equation $\mathbf{L}y = \cos(t)^2$ can be written as a linear combination of functions of the form (4.5) by using the identity $\cos(t)^2 = (1+\cos(2t))/2$. One sees that

$$\mathbf{L}y = \cos(t)^2 = \tfrac{1}{2} + \tfrac{1}{2}\cos(2t)\,.$$

The both terms on the right-hand side above have the form (4.5); the first with $k = 0$, $r = 0$ and $s = 0$, and the second with $k = 0$, $r = 0$ and $s = 2$.

**Example:** The forcing of the equation $\mathbf{L}y = \sin(2t)\cos(3t)$ can be written as a linear combination of functions of the form (4.5) by using the identity

$$\sin(2t)\cos(3t) = \frac{\sin(3t+2t) - \sin(3t-2t)}{2} = \frac{\sin(5t) - \sin(t)}{2}\,.$$

One sees that

$$\mathbf{L}y = \sin(2t)\cos(3t) = \tfrac{1}{2}\sin(5t) - \tfrac{1}{2}\sin(t)\,.$$

The both terms on the right-hand side above have the form (4.5); the first with $k = 0$, $r = 0$ and $s = 5$, and the second with $k = 0$, $r = 0$ and $s = 1$.

**Example:** The forcing of the equation $\mathbf{L}y = \tan(t)$ cannot be written as a linear combination of functions of the form (4.5) because every such function is smooth (infinitely differentiable) while $\tan(t)$ is not defined at $t = \frac{\pi}{2} + m\pi$ for every integer $m$.

The method of undetermined coefficients is based on the observation that, for any forcing of the form

$$\begin{aligned} f(t) = \big(p_0 t^d + p_1 t^{d-1} + \cdots + p_d\big)e^{rt}\cos(st)& \\ + \big(q_0 t^d + q_1 t^{d-1} + \cdots + q_d\big)e^{rt}\sin(st)&\,, \end{aligned} \tag{4.6}$$

for some nonnegative integer $d$ and real numbers $r$ and $s$, one can construct explicit formulas for the particular solution by evaluating at $z = r + is$ the KEY identity and some of its derivatives with respect to $z$. Here we are assuming that $p_0$, $p_1$, $\cdots$, $p_d$ and $q_0$, $q_1$, $\cdots$, $q_d$ are all real and that either $p_0$ or $q_0$ is nonzero. The integer $d$ is the *degree* of the forcing (4.6) while the number $r + is$ is the *characteristic* of the forcing (4.6).

The following observation might help you understand why this is true. Let $P(z)$ denote the characteristic polynomial of $\mathbf{L}$. Then the KEY identity and its first four derivatives are

$$\begin{aligned}
\mathbf{L}e^{zt} &= P(z)e^{zt}\,, \\
\mathbf{L}\big(t\,e^{zt}\big) &= P(z)\,t\,e^{zt} + P'(z)\,e^{zt}\,, \\
\mathbf{L}\big(t^2e^{zt}\big) &= P(z)\,t^2e^{zt} + 2P'(z)\,t\,e^{zt} + P''(z)\,e^{zt}\,, \\
\mathbf{L}\big(t^3e^{zt}\big) &= P(z)\,t^3e^{zt} + 3P'(z)\,t^2e^{zt} + 3P''(z)\,t\,e^{zt} + P'''(z)\,e^{zt}\,, \\
\mathbf{L}\big(t^4e^{zt}\big) &= P(z)\,t^4e^{zt} + 4P'(z)\,t^3e^{zt} + 6P''(z)\,t^2e^{zt} + 4P'''(z)\,t\,e^{zt} + P^{(4)}(z)\,e^{zt}\,.
\end{aligned} \tag{4.7}$$

Notice that when these are evaluated at $z = r + is$ then the terms on the right-hand sides above have the same form as those in the forcing (4.6).

If the characteristic $r+is$ is not a root of $P(z)$ then one needs through the $d^{th}$ derivative of the KEY identity. These should be evaluated at $z = r + is$. A linear combination of the resulting $d + 1$ equations (and their conjugates if $s \neq 0$) can then be found so that its right-hand side equals any $f(t)$ given by (4.6). One then finds a particular solution of the form

$$\begin{aligned}
Y_P(t) = {}&\big(A_0t^d + A_1t^{d-1} + \cdots + A_d\big)e^{rt}\cos(st) \\
&+ \big(B_0t^d + B_1t^{d-1} + \cdots + B_d\big)e^{rt}\sin(st)\,,
\end{aligned} \tag{4.8}$$

where $A_0$, $A_1$, $\cdots$, $A_d$, and $B_0$, $B_1$, $\cdots$, $B_d$ are real constants. Notice that when $s = 0$ the terms involving $B_0$, $B_1$, $\cdots$, $B_d$ all vanish.

More generally, if the characteristic $r + is$ is a root of $P(z)$ of multiplicity $m$ then one needs through the $(m + d)^{th}$ derivative of the KEY identity. These should be evaluated at $z = r + is$. Because $r + is$ is a root of multiplicity $m$, the first $m$ of these will vanish when evaluated at $z = r + is$. A linear combination of the resulting $d + 1$ equations (and their conjugates if $s \neq 0$) can then be found so that its right-hand side equals any $f(t)$ given by (4.6). One then finds a particular solution of the form

$$\begin{aligned}
Y_P(t) = {}&\big(A_0t^{m+d} + A_1t^{m+d-1} + \cdots + A_dt^m\big)e^{rt}\cos(st) \\
&+ \big(B_0t^{m+d} + B_1t^{m+d-1} + \cdots + B_dt^m\big)e^{rt}\sin(st)\,,
\end{aligned} \tag{4.9}$$

where $A_0$, $A_1$, $\cdots$, $A_d$, and $B_0$, $B_1$, $\cdots$, $B_d$ are constants. Notice that when $s = 0$ the terms involving $B_0$, $B_1$, $\cdots$, $B_d$ all vanish. This case includes the previous one if we understand "$r + is$ is a root of $P(z)$ of multiplicity 0" to mean that it is not a root of $P(z)$. For the problems you will face both $m$ and $d$ will be small, so $m + d$ will seldom be larger than 3, and more commonly be 0, 1, or 2.

Given a nonhomogeneous problem $\mathbf{L}y = f(t)$ in which the forcing $f(t)$ is a linear combination of terms that each have the form (4.5), you must first identify the characteristic of each term and group all the terms with the same characteristic together. You then decompose $f(t)$ as

$$f(t) = f_1(t) + f_2(t) + \cdots + f_g(t)\,,$$

where each $f_j(t)$ contains all the terms of a given characteristic. Each $f_j(t)$ will then have the form (4.6) for some characteristic $r + is$ and some degree $d$. You then can apply the method of undetermined coefficients to find particular solutions $Y_{jP}$ to each of

$$\mathbf{L}Y_{1P}(t) = f_1(t)\,, \quad \mathbf{L}Y_{2P}(t) = f_2(t)\,, \quad \cdots \quad \mathbf{L}Y_{gP}(t) = f_g(t)\,. \tag{4.10}$$

Then $Y_P(t) = Y_{1P}(t) + Y_{2P}(t) + \cdots + Y_{gP}(t)$ is a particular solution of $\mathbf{L}y = f(t)$.

**Example:** If $\mathbf{L}y = \mathbf{D}^4 y + 25\mathbf{D}^2 y = f(t)$ with

$$f(t) = e^{2t} + 9\cos(5t) + 3t^2 e^{2t} - 7t^3 \sin(5t) + 6 - 8t\,,$$

you decompose $f(t)$ as $f(t) = f_1(t) + f_2(t) + f_3(t)$, where

$$f_1(t) = \big(1 + 3t^2\big)e^{2t}\,, \quad f_2(t) = 9\cos(5t) - 7t^3\sin(5t)\,, \quad f_3(t) = 6 - 8t\,.$$

Here $f_1(t)$, $f_2(t)$, and $f_3(t)$ contain all the terms of $f(t)$ with characteristic 2, $i5$, and 0 respectively. They each have the form (4.6) with degree 2, 3, and 1 respectively. The characteristic polynomial is $P(z) = z^4 + 25z^2 = z^2(z^2 + 5^2)$, which has roots 0, 0, $-i5$, $i5$. We thereby see that the characteristics 2, $i5$, and 0 have multiplicities 0, 1, and 2 respectively. We can then read off from (4.9) that the method of undertermined coefficients will yield particular solutions for each of the problems in (4.10) that have the forms

$$\begin{aligned}
Y_{1P}(t) &= \big(A_0 t^2 + A_1 t + A_2\big)e^{2t}\,,\\
Y_{2P}(t) &= \big(A_0 t^4 + A_1 t^3 + A_2 t^2 + A_3 t\big)\cos(5t) + \big(B_0 t^4 + B_1 t^3 + B_2 t^2 + B_3 t\big)\sin(5t)\,,\\
Y_{3P}(t) &= A_0 t^3 + A_1 t^2\,.
\end{aligned}$$

Given a nonhomogeneous problem $\mathbf{L}y = f(t)$ in which the forcing $f(t)$ has the form (4.6) with degree $d$ and characteristic $r + is$ that is a root of $P(z)$ of multiplicity $m$, the method of undetermined coefficients will find $Y_P(t)$ in the form (4.9) with $A_0$, $A_1$, $\cdots$, $A_d$, and $B_0$, $B_1$, $\cdots$, $B_d$ as unknowns to be determined. These are the "undetermined coefficients" of the method. There are $2d + 2$ unknowns when $s \neq 0$, and only $d + 1$ unknowns when $s = 0$ because in that case the terms involving $B_0$, $B_1$, $\cdots$, $B_d$ vanish. These unknowns can be determined in one of two ways.

1. **Direct Substitution.** You can substitute the form (4.9) directly into equation (4.3), collect like terms, and match the coefficients in front of each of the linearly independent functions (4.5). In general this leads to a linear algebraic system of either $2d + 2$ equations (if $s \neq 0$) or $d + 1$ equations (if $s = 0$) that must be solved. This is the only approach presented in the book.
2. **KEY Identity Evaluations.** You can evaluate the $m^{th}$ through the $(m + d)^{th}$ derivative of the KEY identity at $z = r + is$, then find a linear combination of the resulting $d + 1$ equations (and their conjugates if $s \neq 0$) whose right-hand side equals any $f(t)$ given by (4.6). This is the approach presented in the lectures.

Both of these approaches always work. They are both fairly painless when $m$ and $d$ are both small and $s = 0$. When $m$ is not small then the first approach is usually faster. When $m$ and $d$ are both small and $s \neq 0$ then the second approach is usually faster. The second approach is also usually faster when the forcing is the sum of terms with different characteristics. They both get long when $d$ is not small. They both will be presented in the following examples.

**Example:** Find a general solution of

$$\mathbf{L}y = \mathbf{D}^2 y + 2\mathbf{D}y + 10y = 6e^{2t}\,, \qquad \text{where} \quad \mathbf{D} = \frac{d}{dt}\,.$$

**Solution:** The characteristic polynomial is

$$P(z) = z^2 + 2z + 10 = (z+1)^2 + 9 = (z+1)^2 + 3^2\,.$$

Its roots are $-1 \pm i3$. Hence,

$$Y_H(t) = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t)\,.$$

To find a particular solution, first notice that the forcing is of the form (4.6) with $d = 0$, $r = 2$, and $s = 0$. Notice that $m = 0$ because the characteristic 2 is not a root of $P(z)$.

**Direct Substitution.** Because $m = d = 0$ and $r + is = 2$, we see from (4.9) that $Y_P$ has the form

$$Y_P(t) = Ae^{2t}\,.$$

Because

$$Y_P'(t) = 2Ae^{2t}\,, \qquad Y_P''(t) = 4Ae^{2t}\,,$$

we see that

$$\begin{aligned} \mathbf{L}Y_P(t) &= Y_P''(t) + 2Y_P'(t) + 10Y_P(t) \\ &= 4Ae^{2t} + 4Ae^{2t} + 10Ae^{2t} = 18Ae^{2t}\,. \end{aligned}$$

If we set $\mathbf{L}Y_P(t) = 6e^{2t}$ then we see that $18A = 6$, whereby $A = \frac{1}{3}$. Hence,

$$Y_P(t) = \tfrac{1}{3}e^{2t}\,.$$

A general solution is therefore

$$y = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) + \tfrac{1}{3}e^{2t}\,.$$

**KEY Identity Evaluations.** Because $m + d = 0$, we will only need the KEY identity:

$$\mathbf{L}e^{zt} = (z^2 + 2z + 10)e^{zt}\,.$$

Evaluate this at $z = 2$ to obtain

$$\mathbf{L}e^{2t} = (4 + 4 + 10)e^{2t} = 18e^{2t}\,.$$

Dividing this by 3 gives

$$\mathbf{L}\big(\tfrac{1}{3}e^{2t}\big) = 6e^{2t}\,,$$

from which we read off that

$$Y_P(t) = \tfrac{1}{3}e^{2t}\,.$$

A general solution is therefore

$$y = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) + \tfrac{1}{3}e^{2t}\,.$$

**Example:** Find a general solution of

$$\mathbf{L}y = \mathbf{D}^2 y + 2\mathbf{D}y + 10y = 4te^{2t}\,, \qquad \text{where} \quad \mathbf{D} = \frac{d}{dt}\,.$$

**Solution:** As before the characteristic polynomial is

$$P(z) = z^2 + 2z + 10 = (z+1)^2 + 3^2\,.$$

Its roots are $-1 \pm i3$. Hence,

$$Y_H(t) = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t)\,.$$

To find a particular solution, first notice that the forcing is of the form (4.6) with $d = 1$, $r = 2$, and $s = 0$. Notice that $m = 0$ because the characteristic 2 is not a root of $P(z)$.

**Direct Substitution.** Because $m = 0$, $d = 1$ and $r + is = 2$, we see from (4.9) that $Y_P$ has the form

$$Y_P(t) = (A_0 t + A_1)e^{2t}\,.$$

Because

$$Y_P'(t) = 2(A_0 t + A_1)e^{2t} + A_0 e^{2t}\,, \qquad Y_P''(t) = 4(A_0 t + A_1)e^{2t} + 4A_0 e^{2t}\,,$$

we see that

$$\begin{aligned}
\mathbf{L}Y_P(t) &= Y_P''(t) + 2Y_P'(t) + 10Y_P(t) \\
&= 4(A_0 t + A_1)e^{2t} + 4A_0 e^{2t} + 4(A_0 t + A_1)e^{2t} + 2A_0 e^{2t} + 10(A_0 t + A_1)e^{2t} \\
&= 18(A_0 t + A_1)e^{2t} + 6A_0 e^{2t} \\
&= 18A_0 te^{2t} + (18A_1 + 6A_0)e^{2t}\,.
\end{aligned}$$

If we set $\mathbf{L}Y_P(t) = 4te^{2t}$ then by equating the coefficients of the linearly independent functions $te^{2t}$ and $e^{2t}$ we see that

$$18A_0 = 4\,, \qquad 18A_1 + 6A_0 = 0\,.$$

Upon solving this linear algebraic system for $A_0$ and $A_1$ we first find that $A_0 = \frac{2}{9}$ and then that $A_1 = -\frac{1}{3}A_0 = -\frac{2}{27}$. Hence,

$$Y_P(t) = \tfrac{2}{9}te^{2t} - \tfrac{2}{27}e^{2t}\,.$$

A general solution is therefore

$$y = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) + \tfrac{2}{9}te^{2t} - \tfrac{2}{27}e^{2t}\,.$$

**KEY Identity Evaluations.** Because $m + d = 1$, we will only need the KEY identity and its first derivative with respect to $z$:

$$\begin{aligned}\mathbf{L}e^{zt} &= (z^2 + 2z + 10)\,e^{zt}\,,\\ \mathbf{L}\big(t\,e^{zt}\big) &= (z^2 + 2z + 10)\,t\,e^{zt} + (2z + 2)\,e^{zt}\,.\end{aligned}$$

Evaluate these at $z = 2$ to obtain

$$\mathbf{L}e^{2t} = 18e^{2t}\,, \qquad \mathbf{L}\big(t\,e^{2t}\big) = 18\,t\,e^{2t} + 6\,e^{2t}\,.$$

Because we want to isolate the $t\,e^{2t}$ term on the right-hand side, subtract one-third the first equation from the second to get

$$\mathbf{L}\big(t\,e^{2t} - \tfrac{1}{3}e^{2t}\big) = \mathbf{L}\big(t\,e^{2t}\big) - \tfrac{1}{3}\mathbf{L}\big(e^{2t}\big) = 18\,t\,e^{2t}\,.$$

After multiplying this by $\frac{2}{9}$ you can read off that

$$Y_P(t) = \tfrac{2}{9}te^{2t} - \tfrac{2}{27}e^{2t}\,.$$

The general solution is therefore

$$y = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) + \tfrac{2}{9}te^{2t} - \tfrac{2}{27}e^{2t}\,.$$

We now illustrate an alternative way to use the KEY Identity Evaluations approach when you have more than one evaluation of the KEY identity and its derivatives, such as in the previous example.

**Example:** Find a general solution of

$$\mathbf{L}y = \mathbf{D}^2y + 2\mathbf{D}y + 10y = 4te^{2t}\,, \qquad \text{where} \quad \mathbf{D} = \frac{d}{dt}\,.$$

**Alternative KEY Identity Evaluations:** Proceed as in the last example up to the point

$$\mathbf{L}e^{2t} = 18e^{2t}\,, \qquad \mathbf{L}\big(t\,e^{2t}\big) = 18\,t\,e^{2t} + 6\,e^{2t}\,.$$

If we set $Y_P(t) = A_0\,t\,e^{2t} + A_1\,e^{2t}$ then we see that

$$\begin{aligned}\mathbf{L}Y_P(t) = A_0\mathbf{L}\big(t\,e^{2t}\big) + A_1\mathbf{L}e^{2t} &= A_0\big(18\,t\,e^{2t} + 6\,e^{2t}\big) + A_1 18e^{2t}\\ &= 18A_0\,t\,e^{2t} + (6A_0 + 18A_1)e^{2t}\,.\end{aligned}$$

If we set $\mathbf{L}Y_P(t) = 4te^{2t}$ then by equating the coefficients of the linearly independent functions $t\,e^{2t}$ and $e^{2t}$ we see that

$$18A_0 - 4\,, \qquad 6A_0 + 18A_1 = 0\,.$$

Upon solving this linear algebraic system for $A_0$ and $A_1$ we first find that $A_0 = \frac{2}{9}$ and then that $A_1 = -\frac{1}{3}A_0 = -\frac{2}{27}$. Hence,

$$Y_P(t) = \tfrac{2}{9}te^{2t} - \tfrac{2}{27}e^{2t}\,.$$

A general solution is therefore

$$y = c_1e^{-t}\cos(3t) + c_2e^{-t}\sin(3t) + \tfrac{2}{9}te^{2t} - \tfrac{2}{27}e^{2t}\,.$$

**Remark:** Notice that this alternative way to using KEY Identity Evaluations led to the same linear algebraic system for $A_0$ and $A_1$ that we got for the Direct Substitution approach. This will generally be the case because they are just two different ways to evaluate $\mathbf{L}Y_P(t)$ for the same family of $Y_P(t)$.

**Example:** Find a general solution of

$$\mathbf{L}y = \mathbf{D}^2y + 2\mathbf{D}y + 10y = \cos(2t)\,, \qquad \text{where} \quad \mathbf{D} = \frac{d}{dt}\,.$$

**Solution:** As before the characteristic polynomial is

$$P(z) = z^2 + 2z + 10 = (z+1)^2 + 3^2\,.$$

Its roots are $-1 \pm i3$. Hence,

$$Y_H(t) = c_1e^{-t}\cos(3t) + c_2e^{-t}\sin(3t)\,.$$

To find a particular solution, first notice that the forcing is of the form (4.6) with $d = 0$, $r = 0$, and $s = 2$. Notice that $m = 0$ because the characteristic $i2$ is not a root of $P(z)$.

**Direct Substitution.** Because $m = d = 0$ and $r + is = i2$, we see from (4.9) that $Y_P$ has the form

$$Y_P(t) = A\cos(2t) + B\sin(2t)\,.$$

Because

$$Y_P'(t) = -2A\sin(2t) + 2B\cos(2t)\,, \qquad Y_P''(t) = -4A\cos(2t) - 4B\sin(2t)\,,$$

we see that

$$\begin{aligned}\mathbf{L}Y_P(t) &= Y_P''(t) + 2Y_P'(t) + 10Y_P(t)\\ &= -4A\cos(2t) - 4B\sin(2t) - 4A\sin(2t) + 4B\cos(2t) + 10A\cos(2t) + 10B\sin(2t)\\ &= (6A + 4B)\cos(2t) + (6B - 4A)\sin(2t)\,.\end{aligned}$$

If we set $\mathbf{L}Y_P(t) = \cos(2t)$ then by equating the coefficients of the linearly independent functions $\cos(2t)$ and $\sin(2t)$ we see that

$$6A + 4B = 1\,, \qquad -4A + 6B = 0\,.$$

Upon solving this system we find that $A = \frac{3}{26}$ and $B = \frac{1}{13}$, whereby

$$Y_P(t) = \tfrac{3}{26}\cos(2t) + \tfrac{1}{13}\sin(2t)\,.$$

A general solution is therefore

$$y = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) + \tfrac{3}{26}\cos(2t) + \tfrac{1}{13}\sin(2t)\,.$$

**KEY Identity Evaluations.** Because $m + d = 0$, we will only need the KEY identity:

$$\mathbf{L}e^{zt} = (z^2 + 2z + 10)e^{zt}\,.$$

Evaluate this at $z = i2$ to obtain

$$\mathbf{L}e^{i2t} = (-4 + i4 + 10)e^{i2t} = (6 + i4)e^{i2t}\,.$$

Dividing this by $6 + i4$ gives

$$\mathbf{L}\left(\frac{1}{6 + i4}\,e^{i2t}\right) = e^{i2t} = \cos(2t) + i\sin(2t)\,.$$

Taking the real part of each side gives

$$\mathbf{L}\left(\mathrm{Re}\left(\frac{1}{6 + i4}\,e^{i2t}\right)\right) = \cos(2t)\,,$$

from which we read off that

$$\begin{aligned}Y_P(t) &= \mathrm{Re}\left(\frac{1}{6 + i4}\,e^{i2t}\right) = \mathrm{Re}\left(\frac{6 - i4}{6^2 + 4^2}\,e^{i2t}\right) = \tfrac{1}{52}\,\mathrm{Re}\big((6 - i4)e^{i2t}\big)\\ &= \tfrac{1}{52}\,\mathrm{Re}\big((6 - i4)(\cos(2t) + i\sin(2t))\big) = \tfrac{6}{52}\cos(2t) + \tfrac{4}{52}\sin(2t)\,.\end{aligned}$$

A general solution is therefore

$$y = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) + \tfrac{3}{26}\cos(2t) + \tfrac{1}{13}\sin(2t)\,.$$

**Example:** Find a general solution of

$$\mathbf{L}y = \mathbf{D}^2 y + 4y = t\,\cos(2t)\,, \qquad \text{where} \quad \mathbf{D} = \frac{d}{dt}\,.$$

**Solution:** This problem has constant coefficients. Its characteristic polynomial is

$$P(z) = z^2 + 4 = z^2 + 2^2\,.$$

Its roots are $\pm i2$. Hence,

$$Y_H(t) = c_1 \cos(2t) + c_2 \sin(2t)\,.$$

To find a particular solution, first notice that the forcing is of the form (4.6) with $d = 1$, $r = 0$, and $s = 2$. Notice that $m = 1$ because the characteristic $i2$ is a root of $P(z)$.

**KEY Identity Evaluations.** Because $m + d = 2$, we will need the first two derivatives of the KEY identity:

$$\begin{aligned}
\mathbf{L}\big(e^{zt}\big) &= (z^2+4)e^{zt}\,,\\
\mathbf{L}\big(t\,e^{zt}\big) &= (z^2+4)t\,e^{zt} + 2z\,e^{zt}\,,\\
\mathbf{L}\big(t^2e^{zt}\big) &= (z^2+4)t^2e^{zt} + 4z\,t\,e^{zt} + 2e^{zt}\,.
\end{aligned}$$

Evaluate these at $z = i2$ to obtain

$$\mathbf{L}\big(e^{i2t}\big) = 0\,, \qquad \mathbf{L}\big(t\,e^{i2t}\big) = i4\,e^{i2t}\,, \qquad \mathbf{L}\big(t^2e^{i2t}\big) = i8\,t\,e^{i2t} + 2e^{i2t}\,.$$

Because $t\,\cos(2t) = \operatorname{Re}(t\,e^{i2t})$, we want to isolate the $t\,e^{i2t}$ term on the right-hand side. This is done by multiplying the second equation by $i\frac{1}{2}$ and adding it to the third to find

$$\mathbf{L}\big((t^2 + i\tfrac{1}{2}t)e^{i2t}\big) = \mathbf{L}\big(t^2e^{i2t}\big) + i\tfrac{1}{2}\mathbf{L}\big(t\,e^{i2t}\big) = i8\,t\,e^{i2t}\,.$$

Now divide this by $i8$ to obtain

$$\mathbf{L}\bigg(\frac{t^2 + i\frac{1}{2}t}{i8}\,e^{i2t}\bigg) = t\,e^{i2t}\,,$$

from which we read off that

$$Y_P(t) = \operatorname{Re}\bigg(\frac{t^2 + i\frac{1}{2}t}{i8}\,e^{i2t}\bigg) = \frac{t}{16}\operatorname{Re}\big((1 - i2t)e^{i2t}\big) = \frac{t}{16}\big(\cos(2t) + 2t\sin(2t)\big)\,.$$

A general solution is therefore

$$y = c_1\cos(2t) + c_2\sin(2t) + \tfrac{1}{16}t\,\cos(2t) + \tfrac{1}{8}t^2\sin(2t)\,.$$

**Direct Substitution.** Because $m = d = 1$ and $r + is = i2$, we see from (4.9) that $Y_P$ has the form

$$Y_P(t) = (A_0t^2 + A_1t)\cos(2t) + (B_0t^2 + B_1t)\sin(2t)\,.$$

Because

$$\begin{aligned}
Y_P'(t) &= -2(A_0t^2 + A_1t)\sin(2t) + (2A_0t + A_1)\cos(2t) \\
&\quad + 2(B_0t^2 + B_1t)\cos(2t) + (2B_0t + B_1)\sin(2t) \\
&= \big(2B_0t^2 + 2(B_1 + A_0)t + A_1\big)\cos(2t) - \big(2A_0t^2 + 2(A_1 - B_0)t - B_1\big)\sin(2t)\,, \\
Y_P''(t) &= -2\big(2B_0t^2 + 2(B_1 + A_0)t + A_1\big)\sin(2t) + \big(4B_0t + 2(B_1 + A_0)\big)\cos(2t) \\
&\quad - 2\big(2A_0t^2 + 2(A_1 - B_0)t - B_1\big)\cos(2t) - \big(4A_0t + 2(A_1 - B_0)\big)\sin(2t) \\
&= -\big(4A_0t^2 + (4A_1 - 8B_0)t - 4B_1 - 2A_0\big)\cos(2t) \\
&\quad - \big(4B_0t^2 + (4B_1 + 8A_0)t + 4A_1 - 2B_0\big)\sin(2t)\,,
\end{aligned}$$

we see that

$$\begin{aligned}
\mathbf{L}Y_P(t) &= Y_P''(t) + 4Y_P(t) \\
&= -\Big[\big(4A_0t^2 + (4A_1 - 8B_0)t - 4B_1 - 2A_0\big)\cos(2t) \\
&\qquad + \big(4B_0t^2 + (4B_1 + 8A_0)t + 4A_1 - 2B_0\big)\sin(2t)\Big] \\
&\quad + 4\Big[(A_0t^2 + A_1t)\cos(2t) + (B_0t^2 + B_1t)\sin(2t)\Big] \\
&= \big(8B_0t + 4B_1 + 2A_0\big)\cos(2t) - \big(8A_0t + 4A_1 - 2B_0\big)\sin(2t)\,.
\end{aligned}$$

If we set $\mathbf{L}Y_P(t) = t\,\cos(2t)$ then by equating the coefficients of the linearly independent functions $\cos(2t)$, $t\,\cos(2t)$, $\sin(2t)$, and $t\,\sin(2t)$, we see that

$$4B_1 + 2A_0 = 0\,, \qquad 8B_0 = 1\,, \qquad 4A_1 - 2B_0 = 0\,, \qquad 8A_0 = 0\,.$$

The solution of this system is $A_0 = 0$, $B_0 = \frac{1}{8}$, $A_1 = \frac{1}{16}$, and $B_1 = 0$, whereby

$$Y_P(t) = \tfrac{1}{16}t\,\cos(2t) + \tfrac{1}{8}t^2\sin(2t)\,.$$

A general solution is therefore

$$y = c_1\cos(2t) + c_2\sin(2t) + \tfrac{1}{16}t\,\cos(2t) + \tfrac{1}{8}t^2\sin(2t)\,.$$

**Remark:** The above example is typical of a case when the KEY identity evaluations approach is far faster than direct substitution. This is because the forcing has a positive degree $d = 1$ and a conjugate pair characteristic, $r \pm is = \pm i2$, of small multiplicity, $m = 1$. This advantage would be much more dramatic for larger $d$, but would diminish some for larger $m$.

**4.3: Variation of Parameters.** We now return to the discussion of the general case

$$\mathbf{L}(t)y = f(t)\,. \tag{4.11}$$

> **Important Fact:** If you know a general solution of the associated homogeneous equation $\mathbf{L}(t)y = 0$ then you can *always* reduce the construction of a general solution of (4.11) to the problem of finding $n$ primitives. The method for doing this is called *variation of parameters.*

Because at this point you only know how to find general solutions of homogeneous equations with constant coefficients, problems you will be given will generally fall into one of two categories. Either (1) the operator $\mathbf{L}(t)$ will have variable coefficients and you will be given a fundamental set of solutions for the associated homogeneous equation, or (2) the operator $\mathbf{L}(t)$ will have constant coefficients (i.e. $\mathbf{L}(t) = \mathbf{L}$) and you will be expected to find a fundamental set of solutions for the associated homogeneous equation.

We will now derive the method of variation of parameters for second order equations in the normal form

$$\mathbf{L}(t)y = \frac{d^2y}{dt^2} + a_1(t)\frac{dy}{dt} + a_2(t)y = f(t)\,. \tag{4.12}$$

Suppose you know that $Y_1(t)$ and $Y_2(t)$ are linearly independent solutions of the associated homogeneous equation $\mathbf{L}(t)y = 0$. A general solution of the associated homogeneous equation is therefore given by

$$Y_H(t) = c_1Y_1(t) + c_2Y_2(t)\,. \tag{4.13}$$

The idea of the method of variation of parameters is to seek solutions of (4.12) in the form

$$y = u_1(t)Y_1(t) + u_2(t)Y_2(t)\,. \tag{4.14}$$

In other words you simply replace the arbitrary constants $c_1$ and $c_2$ in (4.13) with unknown functions $u_1(t)$ and $u_2(t)$. These functions are the varying parameters referred to in the title of the method. These two functions will be governed by a system of two equations, one of which is derived by requiring that (4.12) is satisfied, and the other of which is chosen to simplify the resulting system.

Let us see how this is done. Differentiating (4.14) yields

$$\frac{dy}{dt} = u_1(t)Y_1'(t) + u_2(t)Y_2'(t) + u_1'(t)Y_1(t) + u_2'(t)Y_2(t)\,. \tag{4.15}$$

We now choose to impose the condition

$$u_1'(t)Y_1(t) + u_2'(t)Y_2(t) = 0\,, \tag{4.16}$$

whereby (4.15) simplifies to

$$\frac{dy}{dt} = u_1(t)Y_1'(t) + u_2(t)Y_2'(t)\,. \tag{4.17}$$

Differentiating (4.17) then yields

$$\frac{d^2y}{dt^2} = u_1(t)Y_1''(t) + u_2(t)Y_2''(t) + u_1'(t)Y_1'(t) + u_2'(t)Y_2'(t)\,. \tag{4.18}$$

Now substituting (4.14), (4.17), and (4.18) into (4.12), grouping the terms that multiply $u_1(t)$, $u_1'(t)$, $u_2(t)$, and $u_2'(t)$, and using the fact that $\mathbf{L}(t)Y_1(t) = 0$ and $\mathbf{L}(t)Y_2(t) = 0$, we obtain

$$\begin{aligned}
f(t) = \mathbf{L}(t)y &= \frac{d^2y}{dt^2} + a_1(t)\frac{dy}{dt} + a_2(t)y \\
&= \big[u_1(t)Y_1''(t) + u_2(t)Y_2''(t) + u_1'(t)Y_1'(t) + u_2'(t)Y_2'(t)\big] \\
&\quad + a_1(t)\big[u_1(t)Y_1'(t) + u_2(t)Y_2'(t)\big] + a_2(t)\big[u_1(t)Y_1(t) + u_2(t)Y_2(t)\big] \\
&= u_1(t)\big[Y_1''(t) + a_1(t)Y_1'(t) + a_2(t)Y_1(t)\big] + u_1'(t)Y_1'(t) \\
&\quad + u_2(t)\big[Y_2''(t) + a_1(t)Y_2'(t) + a_2(t)Y_2(t)\big] + u_2'(t)Y_2'(t) \\
&= u_1(t)\big[\mathbf{L}(t)Y_1(t)\big] + u_1'(t)Y_1'(t) + u_2(t)\big[\mathbf{L}(t)Y_2(t)\big] + u_2'(t)Y_2'(t) \\
&= u_1'(t)Y_1'(t) + u_2'(t)Y_2'(t)\,.
\end{aligned} \tag{4.19}$$

The resulting system that governs $u_1(t)$ and $u_2(t)$ is thereby given by (4.16) and (4.19):

$$\begin{aligned}
u_1'(t)Y_1(t) + u_2'(t)Y_2(t) &= 0\,, \\
u_1'(t)Y_1'(t) + u_2'(t)Y_2'(t) &= f(t)\,.
\end{aligned} \tag{4.20}$$

This is a linear system of two algebraic equations for $u_1'(t)$ and $u_2'(t)$. Because

$$\big(Y_1(t)Y_2'(t) - Y_2(t)Y_1'(t)\big) = W[Y_1, Y_2](t) \neq 0\,,$$

one can always solve this system to find

$$u_1'(t) = -\frac{Y_2(t)f(t)}{W[Y_1, Y_2](t)}\,, \qquad u_2'(t) = \frac{Y_1(t)f(t)}{W[Y_1, Y_2](t)}\,.$$

Letting $u_{1P}(t)$ and $u_{2P}(t)$ be any primitives of the respective right-hand sides above, one sees that

$$u_1(t) = c_1 + u_{1P}(t)\,, \qquad u_2(t) = c_2 + u_{2P}(t)\,,$$

whereby (4.14) yields the general solution

$$y = c_1Y_1(t) + u_{1P}(t)Y_1(t) + c_2Y_2(t) + u_{2P}(t)Y_2(t)\,.$$

Notice that this decomposes as $y = Y_H(t) + Y_P(t)$ where

$$Y_H(t) = c_1 Y_1(t) + c_2 Y_2(t)\,, \qquad Y_P(t) = u_{1P}(t) Y_1(t) + u_{2P}(t) Y_2(t)\,.$$

The best way to apply this method in practice is not to memorize one of the various formulas for the final solution given in the book, but rather to construct the linear system (4.20), which can then be rather easily solved for $u_1'(t)$ and $u_2'(t)$. Given $Y_1(t)$ and $Y_2(t)$, a fundamental set of solutions to the associated homogeneous equation, you proceed as follows.

1) Write the form of the solution you seek:

$$y = u_1(t) Y_1(t) + u_2(t) Y_2(t)\,.$$

2) Write the linear algebraic system for $u_1'(t)$ and $u_2'(t)$:

$$\begin{aligned} u_1'(t) Y_1(t) + u_2'(t) Y_2(t) &= 0\,, \\ u_1'(t) Y_1'(t) + u_2'(t) Y_2'(t) &= f(t)\,. \end{aligned}$$

   The form of the left-hand sides of this system mimics the form of the solution we seek. The first equation simply replaces $u_1(t)$ and $u_2(t)$ with $u_1'(t)$ and $u_2'(t)$, while the second also replaces $Y_1(t)$ and $Y_2(t)$ with $Y_1'(t)$ and $Y_2'(t)$. The $f(t)$ on the right-hand side will be correct only if you have written the equation $\mathbf{L}(t)y = f(t)$ in normal form!

3) Solve the linear algebraic system to find explicit expressions for $u_1'(t)$ and $u_2'(t)$. This is always very easy to do, especially if you start with the first equation.

4) Find primitives $u_{1P}(t)$ and $u_{2P}(t)$ of these expressions. If you cannot find a primitive analytically then express that primitive in terms of a definite integral. One then has

$$u_1(t) = c_1 + u_{1P}(t)\,, \qquad u_2(t) = c_2 + u_{2P}(t)\,,$$

   where $c_1$ and $c_2$ are the arbitrary constants of integration.

5) Upon placing this result into the form of the solution that you wrote down in step 1, you will obtain the general solution

$$y = Y_H(t) + Y_P(t)\,,$$

   where

$$Y_H(t) = c_1 Y_1(t) + c_2 Y_2(t)\,, \qquad Y_P(t) = u_{1P}(t) Y_1(t) + u_{2P}(t) Y_2(t)\,.$$

   If the problem is an initial-value problem you must determine $c_1$ and $c_2$ from the initial conditions.

**Example:** Find a general solution of

$$\frac{d^2y}{dt^2} + y = \sec(t)\,.$$

Before presenting the solution, notice that while this equation has constant coefficients, the driving is not of the form that would allow you to use the method of undetermined coefficients. You should be able to recognize this right away and thereby see that the only method you can use to solve this problem is variation of parameters.

**Solution:** Because this problem has constant coefficients, it is easily found that

$$Y_H(t) = c_1 \cos(t) + c_2 \sin(t)\,.$$

Hence, we will seek a solution of the form

$$y = u_1(t)\cos(t) + u_2(t)\sin(t)\,,$$

where

$$\begin{aligned} u_1'(t)\cos(t) + u_2'(t)\sin(t) &= 0\,, \\ -u_1'(t)\sin(t) + u_2'(t)\cos(t) &= \sec(t)\,. \end{aligned}$$

Solving this system by any means you choose yields

$$u_1'(t) = -\frac{\sin(t)}{\cos(t)}\,, \qquad u_2'(t) = 1\,.$$

These can be integrated analytically to obtain

$$u_1(t) = c_1 + \log(|\cos(t)|)\,, \qquad u_2(t) = c_2 + t\,.$$

Therefore a general solution is

$$y = c_1 \cos(t) + c_2 \sin(t) + \log(|\cos(t)|)\cos(t) + t\sin(t)\,.$$

**Example:** Given that $t$ and $t^2 - 1$ are a fundamental set of solutions of the associated homogeneous equation, find a general solution of

$$(1+t^2)\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + 2y = (1+t^2)^2 e^t\,.$$

Before presenting the solution, you should be able to recognize that this equation has nonconstant coefficients, and thereby see that the only method you can use to solve this problem is variation of parameters. You should also notice that this equation is not in normal form, so you should bring it into the normal form

$$\frac{d^2y}{dt^2} - \frac{2t}{1+t^2}\,\frac{dy}{dt} + \frac{2}{1+t^2}\,y = (1+t^2)e^t\,.$$

**Solution:** Because $t$ and $t^2-1$ are a fundamental set of solutions of the associated homogeneous equation, we have

$$Y_H(t) = c_1 t + c_2(t^2-1)\,.$$

Hence, we will seek a solution of the form

$$y = u_1(t)t + u_2(t)(t^2-1)\,,$$

where

$$\begin{aligned} u_1'(t)t + u_2'(t)(t^2-1) &= 0\,,\\ u_1'(t)1 + u_2'(t)2t &= (1+t^2)e^t\,. \end{aligned}$$

Solving this system by any means you choose yields

$$u_1'(t) = -(t^2-1)e^t\,, \qquad u_2'(t) = te^t\,.$$

These can be integrated analytically "by parts" to obtain

$$u_1(t) = c_1 - (t-1)^2e^t\,, \qquad u_2(t) = c_2 + (t-1)e^t\,.$$

Therefore a general solution is

$$\begin{aligned} y &= c_1 t + c_2(t^2-1) - (t-1)^2e^t t + (t-1)e^t(t^2-1)\\ &= c_1 t + c_2(t^2-1) - (t-1)^2e^t\,. \end{aligned}$$

**Remark:** The method of variation of parameters extends to higher order linear equations in the normal form

$$\mathbf{L}(t)y = \frac{d^ny}{dt^n} + a_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_{n-1}(t)\frac{dy}{dt} + a_n(t)y = f(t)\,. \tag{4.21}$$

While this material was not covered in class and you will not be tested on it, a summary is given here for the sake of completeness.

Suppose you know that $Y_1(t), Y_2(t), \cdots, Y_n(t)$ are linearly independent solutions of the associated homogeneous equation $\mathbf{L}(t)y = 0$. The general solution of the associated homogeneous equation is therefore given by

$$Y_H(t) = c_1Y_1(t) + c_2Y_2(t) + \cdots + c_nY_n(t)\,. \tag{4.22}$$

The idea of the method of variation of parameters is to seek solutions of (4.21) in the form

$$y = u_1(t)Y_1(t) + u_2(t)Y_2(t) + \cdots + u_n(t)Y_n(t)\,, \tag{4.23}$$

where $u_1'(t), u_2'(t), \cdots, u_n'(t)$ satisfy the linear algebraic system

$$\begin{aligned}
u_1'(t)Y_1(t) \quad &+ \quad u_2'(t)Y_2(t) \quad + \cdots + \quad u_n'(t)Y_n(t) &&= 0\,,\\
u_1'(t)Y_1'(t) \quad &+ \quad u_2'(t)Y_2'(t) \quad + \cdots + \quad u_n'(t)Y_n'(t) &&= 0\,,\\
&&&\vdots\\
u_1'(t)Y_1^{(n-2)}(t) &+ u_2'(t)Y_2^{(n-2)}(t) + \cdots + u_n'(t)Y_n^{(n-2)}(t) &&= 0\,,\\
u_1'(t)Y_1^{(n-1)}(t) &+ u_2'(t)Y_2^{(n-1)}(t) + \cdots + u_n'(t)Y_n^{(n-1)}(t) &&= f(t)\,.
\end{aligned} \tag{4.24}$$

Because

$$\det\begin{pmatrix} Y_1(t) & Y_2(t) & \cdots & Y_n(t)\\ Y_1'(t) & Y_2'(t) & \cdots & Y_n'(t)\\ \vdots & \vdots & \ddots & \vdots\\ Y_1^{(n-1)}(t) & Y_2^{(n-1)}(t) & \cdots & Y_n^{(n-1)}(t)\end{pmatrix} = W[Y_1, Y_2, \cdots, Y_n](t) \neq 0\,,$$

the linear algebraic system (4.24) may be solved (by any method you choose) to find explicit expressions for $u_1'(t)$, $u_2'(t)$, $\cdots$, $u_n'(t)$. For example, when $n = 3$ you find

$$u_1'(t) = \frac{W[Y_2, Y_3](t)f(t)}{W[Y_1, Y_2, Y_3](t)}\,, \quad u_2'(t) = \frac{W[Y_3, Y_1](t)f(t)}{W[Y_1, Y_2, Y_3](t)}\,, \quad u_3'(t) = \frac{W[Y_1, Y_2](t)f(t)}{W[Y_1, Y_2, Y_3](t)}\,.$$

Find primitives $u_{1P}(t)$, $u_{2P}(t)$, $\cdots$, $u_{nP}(t)$ of these expressions. If you cannot find a primitive analytically then express that primitive in terms of a definite integral. One then has

$$u_1(t) = c_1 + u_{1P}(t)\,, \quad u_2(t) = c_2 + u_{2P}(t)\,, \quad \cdots \quad u_n(t) = c_n + u_{nP}(t)\,,$$

where $c_1, c_2, \cdots, c_n$ are the arbitrary constants of integration. The general solution given by (4.23) is therefore

$$y = Y_H(t) + Y_P(t)\,,$$

where

$$\begin{aligned}
Y_H(t) &= c_1Y_1(t) + c_2Y_2(t) + \cdots + c_nY_n(t)\,,\\
Y_P(t) &= u_{1P}(t)Y_1(t) + u_{2P}(t)Y_2(t) + \cdots + u_{nP}(t)Y_n(t)\,.
\end{aligned}$$

If the problem is an initial-value problem you must determine $c_1, c_2, \cdots, c_n$ from the initial conditions.

## Appendix A. Linear Algebraic Systems and Determinants

**A.1: Linear Algebraic Systems.** In all of the examples in section 2.1 we were able to find a unique solution $c_1$, $c_2$, $\cdots$, $c_n$ to the linear algebraic system (2.2) which enabled us to solve the general initial value problem for any choice of initial data $y_0$, $y_1$, $\cdots$, $y_{n-1}$. In this section we will characterize when a linear algebraic system always has a unique solution.

A linear algebric system of $n$ equations for $n$ unknowns $x_1$, $x_2$, $\cdots$, $x_n$ has the general form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1\,, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2\,, \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n\,. \end{aligned} \tag{A.1}$$

Here the $n^2$ numbers $\{a_{ij} : i, j = 1, 2, \cdots, n\}$ are called the *coefficients* of the system and the $n$ numbers $b_1$, $b_2$, $\cdots$, $b_n$ are called the *forcing.* There are two questions regarding the existence of solutions that we want to address. The first is:

*When does system (A.1) have a unique solution for every forcing* $b_1$*,* $b_2$*,* $\cdots$*,* $b_n$*?*

The second is:

*When does system (A.1) with* $b_1 = b_2 = \cdots = b_n = 0$ *have a nonzero solution?*

Here "nonzero solution" means that it is a solution with $x_k \neq 0$ for some index $k$ — i.e. it is a solution that is not the "trivial solution" $x_1 = x_2 = \cdots = x_n = 0$.

These questions are clearly related. Let us suppose that system (A.1) has two different solutions for some set of numbers $b_1$, $b_2$, $\cdots$, $b_n$. We denote one of these solutions by $x_1$, $x_2$, $\cdots$, $x_n$ and the other by $y_1$, $y_2$, $\cdots$, $y_n$. Set

$$z_1 = x_1 - y_1\,, \qquad z_2 = x_2 - y_2\,, \qquad \cdots, \qquad z_n = x_n - y_n\,.$$

Then one can show that $z_1$, $z_2$, $\cdots$, $z_n$ is a nonzero solution of

$$\begin{aligned} a_{11}z_1 + a_{12}z_2 + \cdots + a_{1n}z_n &= 0\,, \\ a_{21}z_1 + a_{22}z_2 + \cdots + a_{2n}z_n &= 0\,, \\ &\vdots \\ a_{n1}z_1 + a_{n2}z_2 + \cdots + a_{nn}z_n &= 0\,. \end{aligned} \tag{A.2}$$

But this is system (A.1) with $b_1 = b_2 = \cdots = b_n = 0$.

Conversely, if system (A.2) has a nonzero solution $z_1$, $z_2$, $\cdots$, $z_n$ then no solution of (A.1) is unique for any forcing $b_1$, $b_2$, $\cdots$, $b_n$. To see this, let $y_1$, $y_2$, $\cdots$, $y_n$ be a solution of (A.1) for some forcing $b_1$, $b_2$, $\cdots$, $b_n$:

$$\begin{aligned} a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n &= b_1 \,, \\ a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n &= b_2 \,, \\ &\vdots \\ a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n &= b_n \,. \end{aligned}$$

For any nonzero real number $\alpha$ set

$$x_1 = y_1 + \alpha z_1 \,, \qquad x_2 = y_2 + \alpha z_2 \,, \qquad \cdots \,, \qquad x_n = y_n + \alpha z_n \,.$$

Then one can show that $x_1$, $x_2$, $\cdots$, $x_n$ is a solution of system (A.1) for the same forcing $b_1$, $b_2$, $\cdots$, $b_n$ that is different than $y_1$, $y_2$, $\cdots$, $y_n$.

**A.2: Determinants.** Answers to our questions can depend only on the coefficients $\{a_{ij} \,:\, i,j = 1, 2, \cdots, n\}$. It is helpful to write these coefficients as an $n{\times}n$ matrix $A$:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} .$$

The answers will be given in terms of a quantity $\det(A)$, called the *determinant* of $A$. For $n = 1$, 2, and 3 the quantity $\det(A)$ is given by

$$\begin{aligned} \det(\, a_{11} \,) &= a_{11} \,, \\ \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} &= a_{11}a_{22} - a_{12}a_{21} \,, \\ \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \,. \end{aligned} \tag{A.3}$$

The best way to remember these formulas is visually. The formula for the 2×2 determinant can be remembered as the product of the terms on the ↘ diagonal minus the product of the terms on the ↗ diagonal. (Draw these two diagonal arrows on the above 2×2 matrix as was done in class.) The formula for the 3×3 determinant can be remembered by first augmenting the matrix by repeating the first two columns, thereby creating the 3×5 augmented matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{pmatrix} .$$

The formula is then the sum of the products of the terms on the $\searrow\searrow\searrow$ diagonals minus the sum of the products of the terms on the $\nearrow\nearrow\nearrow$ diagonals. (Draw these six diagonal arrows on the above 3×5 augmented matrix as was done in class.) You can also use the "spaghetti" drawing on the 3×3 matrix.

In general the determinant of the $n\times n$ matrix $A$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

can be expanded in terms of the determinants of $(n-1)\times(n-1)$ matrices $A_{jk}$ as

$$\det(A) = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} \det(A_{jk}) \,, \tag{A.4}$$

where $A_{jk}$ denotes the $(n-1)\times(n-1)$ matrix obtained by crossing out the $j^{th}$ row and $k^{th}$ column of $A$. This is called the *Laplace formula.* For example, for $n = 2$ and $j = 1$ this formula gives

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}\det(a_{22}) - a_{12}\det(a_{21}) = a_{11}a_{22} - a_{12}a_{21} \,.$$

For $n = 3$ and $j = 1$ this formula gives

$$\begin{aligned} \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} &= a_{11}\det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12}\det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13}\det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \\ &= a_{11}\big(a_{22}a_{33} - a_{23}a_{32}\big) - a_{12}\big(a_{21}a_{33} - a_{23}a_{31}\big) \\ &\qquad + a_{13}\big(a_{21}a_{32} - a_{22}a_{31}\big) \,. \end{aligned}$$

For $n = 4$ and $j = 1$ this formula gives

$$\begin{aligned} \det\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} &= a_{11}\det\begin{pmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{pmatrix} - a_{12}\det\begin{pmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{pmatrix} \\ &\quad + a_{13}\det\begin{pmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{pmatrix} - a_{14}\det\begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix} \,. \end{aligned}$$

Fully expanded, the formula for the $n\times n$ determinant is the sum of $n!$ products. There is no simple "diagonal" picture that can be used to remember it visually when $n > 3$. However the Laplace formula (A.4) works well provided $n$ is not too large.

**Exercise:** Prove the following evaluation of the $n \times n$ determinant

$$\det\begin{pmatrix} 1 & 0 & \cdots & 0 \\ a_{21} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_{n1} & \cdots & a_{n(n-1)} & 1 \end{pmatrix} = 1 \,.$$

Hint: Use the Laplace formula and induction on $n$.

**A.3: Existence of Solutions.** The answers to our questions are as follows.

> **Theorem A.1:** System (A.1) has a unique solution for every forcing $b_1$, $b_2$, $\cdots$, $b_n$ if and only if $\det(A) \neq 0$.
>
> **Theorem A.2:** System (A.2) has a nonzero solution if and only if $\det(A) = 0$.

**Remark:** Half of Theorem A.2 is implied by Theorem A.1. Indeed, Theorem A.1 implies that if $\det(A) \neq 0$ then the trivial solution $z_1 = z_2 = \cdots = z_n = 0$ is the only solution of system (A.2). Hence, once Theorem A.1 is established, all that one needs to show to establish Theorem A.2 is that if $\det(A) = 0$ then system (A.2) has a nonzero solution.

We will not give proofs of Theorem A.1 and Theorem A.2 for general $n$ because they are beyond the scope of this course. They are covered in sufficiently advanced linear algebra courses. However, we will give proofs of these theorems for the cases $n = 1$ and $n = 2$. While you will not be expected to know these proofs, you will be expected to know both theorems.

**Proofs:** Of course, when $n = 1$ system (A.1) is simply the single equation

$$a_{11}x_1 = b_1 \,.$$

This clearly has the unique solution

$$x_1 = \frac{b_1}{a_{11}} \,,$$

if and only if $a_{11} \neq 0$. Hence, Theorem A.1 holds for $n = 1$.

When $n = 1$ system (A.2) is simply the single equation

$$a_{11}z_1 = 0 \,.$$

Clearly, if $a_{11} = 0$ then any $z_1$ satisfies this equation. Hence, Theorem A.2 holds for $n = 1$.

When $n = 2$ system (A.1) is the two equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \,, \\ a_{21}x_1 + a_{22}x_2 &= b_2 \,. \end{aligned} \tag{A.5}$$

First eliminate $x_2$ by multiplying the first equation in (A.5) by $a_{22}$, the second by $a_{12}$, and then subtracting the results to obtain

$$\big(a_{11}a_{22} - a_{12}a_{21}\big)x_1 = b_1a_{22} - b_2a_{12}\,. \tag{A.6a}$$

Similarly, eliminate $x_1$ by multiplying the second equation in (A.5) by $a_{11}$, the second by $a_{21}$, and then subtracting the results to obtain

$$\big(a_{11}a_{22} - a_{12}a_{21}\big)x_2 = b_2a_{11} - b_1a_{21}\,. \tag{A.6b}$$

If $(a_{11}a_{22} - a_{12}a_{21}) \neq 0$ then for any choice of $b_1$ and $b_2$ the system (A.5) has the unique solution

$$x_1 = \frac{b_1a_{22} - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}\,, \qquad x_2 = \frac{a_{11}b_2 - b_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}}\,,$$

On the other hand, if $(a_{11}a_{22} - a_{12}a_{21}) = 0$ then by (A.6) there can only be a solution of (A.5) when

$$b_1a_{22} - b_2a_{12} = 0\,, \qquad \text{and} \qquad b_2a_{11} - b_1a_{21} = 0\,.$$

But unless $a_{11} = a_{12} = a_{12} = a_{12} = 0$ one can always find values of $b_1$ and $b_2$ for which at least one of these quantities is nonzero. And if $a_{11} = a_{12} = a_{12} = a_{12} = 0$ then it is clear that that (A.5) only has a solution when $b_1 = b_2 = 0$. Hence, if $(a_{11}a_{22} - a_{12}a_{21}) = 0$ we can always find values of $b_1$ and $b_2$ for which system (A.3) has no solution. We conclude that system (A.5) has a unique solution for every choice of $b_1$ and $b_2$ if and only if $(a_{11}a_{22} - a_{12}a_{21}) \neq 0$. This proves Theorem A.1 for $n = 2$.

When $n = 2$ the system (A.2) is the two equations

$$\begin{aligned} a_{11}z_1 + a_{12}z_2 &= 0\,, \\ a_{21}z_1 + a_{22}z_2 &= 0\,. \end{aligned} \tag{A.7}$$

If $(a_{11}a_{22} - a_{12}a_{21}) = 0$ then both

$$z_1 = a_{22}\,, \quad z_2 = -a_{21}\,, \qquad \text{and} \qquad z_1 = -a_{12}\,, \quad z_2 = a_{11}\,,$$

give solutions of (A.7), at least one of which will be nonzero unless $a_{11} = a_{12} = a_{12} = a_{12} = 0$. However, when $a_{11} = a_{12} = a_{12} = a_{12} = 0$ then any values of $z_1$ and $z_2$ satisfy (A.7). Hence, (A.7) has a nonzero solution in either case. Therefore Theorem A.2 holds for $n = 2$. $\square$