Quiz 2 Solutions, Math 246, Professor David Levermore
Wednesday, 13 February 2008

(1) [2] What is the interval of existence for the solution of the initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} + \tan(t)\, y = \cos(t)\,, \qquad y(\pi) = 0\,.$$

(You do not have to find the solution!)

Solution: The interval of existence is $(\frac{\pi}{2}, \frac{3\pi}{2})$, the endpoints of which are the singularities of $\tan(t)$ that bracket the initial time π.

(2) [3] Sketch the phase-line for the initial-value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(4 - x^2)\,, \qquad x(0) = x_o\,.$$

Identify the stationary (equilibrium) points as either stable, unstable, or semistable. (You do not have to find the solution!)

Solution: Because $x(4 - x^2) = x(2 + x)(2 - x)$, the phase-line is

$$\begin{array}{ccccccc} + & & - & & + & & - \\ \rightarrow\rightarrow\rightarrow\rightarrow & \bullet & \leftarrow\leftarrow\leftarrow\leftarrow & \bullet & \rightarrow\rightarrow\rightarrow\rightarrow & \bullet & \leftarrow\leftarrow\leftarrow\leftarrow \\ & -2 & & 0 & & 2 & \\ & \text{stable} & & \text{unstable} & & \text{stable} & \end{array}$$

(3) [2] Consider the initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} + \tfrac{1}{2}y^3 = 0\,, \qquad y(0) = y_o \quad (y_o \neq 0)\,.$$

The solution satisfies

$$\frac{1}{y^2} = t + \frac{1}{y_o^2}\,.$$

Find the interval of existence for the solution. (You do not have to find the explicit solution!)

Solution: The interval of existence is $(-1/y_o^2, \infty)$ because one needs

$$t + \frac{1}{y_o^2} > 0\,.$$

(4) [3] A population of mosquitoes in a certain area increases at a rate proportional to the current population, and in the absence of other factors, the population doubles each week. There are 200,000 mosquitoes in the area initially, and predators eat 140,000 mosquitoes per week. Write down an initial-value problem that governs the population of mosquitoes in the area at any time. (You do not have to solve the initial-value problem!)

Solution: Let $M(t)$ be the number of mosquitoes at time t weeks. Doubling each week implies a growth rate of $\log(2)$. The initial-value problem that M satisfies is then

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \log(2)M - 140,000\,, \qquad M(0) = 200,000\,.$$