Quiz 4 Solutions, Math 246, Professor David Levermore
Wednesday, 5 March 2008

(1) [4] What is the interval of existence for the solution of the following initial value problems.

(a) $(4t - t^2)\dfrac{d^2y}{dt^2} + (1 + t^2)\dfrac{dy}{dt} + \cos(t)y = \sin(t)\,, \quad y(5) = 1\,, \quad y'(5) = 0\,.$

Solution: The linear normal form is $\dfrac{d^2y}{dt^2} + \dfrac{1+t^2}{t(4-t)}\dfrac{dy}{dt} + \dfrac{\cos(t)}{t(4-t)}y = \dfrac{\cos(t)}{t(4-t)}$. The coefficients and forcing are defined and continuous everywhere except $t = 0$ and $t = 4$. The initial time is $t = 5$. The interval of existence is therefore $t > 4$.

(b) $\dfrac{d^4x}{dt^4} + x = \tan(t)\,, \quad x(\pi) = x'(\pi) = x''(\pi) = x'''(\pi) = 0\,.$

Solution: The equation is already in linear normal form. The coefficients and forcing are defined and continuous everywhere except $t = k\pi + \frac{\pi}{2}$ for some integer k. The initial time is $t = \pi$. The interval of existence is therefore $\frac{\pi}{2} < t < \frac{3\pi}{2}$.

(2) [3] Compute the Wronskian of the functions $Y_1(t) = \cos(3t)$ and $Y_2(t) = \sin(3t)$. (Evaluate the determinant and simplify.)

Solution: Because $Y_1'(t) = -3\sin(3t)$ and $Y_2'(t) = 3\cos(3t)$, the Wronskian is

$$\begin{aligned} W[Y_1, Y_2](t) &= \det\begin{pmatrix} Y_1(t) & Y_2(t) \\ Y_1'(t) & Y_2'(t) \end{pmatrix} = \det\begin{pmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{pmatrix} \\ &= 3\cos(3t)^2 + 3\sin(3t)^2 = 3\,. \end{aligned}$$

(3) [3] Given that $\cos(3t)$ and $\sin(3t)$ are solutions of $\dfrac{d^2y}{dt^2} + 9y = 0$, find a solution $Y(t)$ that satisfies the initial conditions $Y(0) = 3$, $Y'(0) = 6$.

Solution: Let $Y(t) = c_1\cos(3t) + c_2\sin(3t)$. Then $Y'(t) = -3c_1\sin(3t) + 3c_2\cos(3t)$. To satisfy the initial conditions one needs

$$3 = Y(0) = c_1\,, \qquad 6 = Y'(0) = 3c_2\,.$$

Hence, $c_1 = 3$ and $c_2 = 2$. The solution of the initial value problem is therefore

$$Y(t) = 3\cos(3t) + 2\sin(3t)\,.$$