Quiz 5 Solutions, Math 246, Professor David Levermore
Wednesday, 12 March 2008

(1) [6] Give a general solution to each of the following.

(a) $\dfrac{d^3y}{dt^3} + 4\dfrac{d^2y}{dt^2} + 13\dfrac{dy}{dt} = 0$

Solution: The coefficients are constant. The characteristic polynomial is

$$\begin{aligned} p(z) &= z^3 + 4z^2 + 13z = z(z^2 + 4z + 13) = z\big((z+2)^2 + 9\big) \\ &= z\big((z+2)^2 + 3^2\big) . \end{aligned}$$

Its roots are $z = 0, -2 + i3, -2 - i3$. A general solution is therefore

$$c_1 + c_2 e^{-2t} \cos(3t) + c_3 e^{-2t} \sin(3t) .$$

(b) $\dfrac{d^4x}{dt^4} - x = 0$

Solution: The coefficients are constant. The characteristic polynomial is

$$\begin{aligned} p(z) &= z^4 - 1 = (z^2 - 1)(z^2 + 1) = (z - 1)(z + 1)(z^2 + 1) \\ &= (z - 1)(z + 1)\big(z^2 + 1^2\big) . \end{aligned}$$

Its roots are $z = 1, -1, i, -i$. A general solution is therefore

$$c_1 e^t + c_2 e^{-t} + c_3 \cos(t) + c_4 \sin(t) .$$

(2) [4] A seventh order linear differential operator with constant coefficients has a characteristic polynomial $p(z)$ with roots $z = 5, 5, 5, 3 + 2i, 3 + 2i, 3 - 2i, 3 - 2i$. Give its general solution.

Solution: A general solution is

$$c_1 e^{5t} + c_2 t\, e^{5t} + c_3 t^2 e^{5t} + c_4 e^{3t} \cos(2t) + c_5 e^{3t} \sin(2t) + c_6 t\, e^{3t} \cos(2t) + c_7 t\, e^{3t} \sin(2t) .$$

The real root 5 with multiplicity 3 contributes

$$e^{5t} , \qquad t\, e^{5t} , \qquad t^2 e^{5t} .$$

The complex conjugate pair of roots $3 \pm i2$ with multiplicity 2 contributes

$$e^{3t} \cos(2t) , \qquad e^{3t} \sin(2t) , \qquad t\, e^{3t} \cos(2t) , \qquad t\, e^{3t} \sin(2t) .$$