Quiz 6 Solutions, Math 246, Professor David Levermore
Wednesday, 26 March 2008

(1) [5] Find a general solution of

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - 3x = e^t - 3t$$

Solution. The characteristic polynomial is $P(z) = z^2 + 2z - 3 = (z-1)(z+3)$ and has roots 1 and -3. The solution of the associated homogeneous problem is

$$Y_H(t) = c_1 e^t + c_2 e^{-3t}\,.$$

The forcing term e^t has degree 0 and characteristic 1 which is a root of $P(z)$ of multiplicity 1. The forcing term $-3t$ has degree 1 and characteristic 0 which is a root of $P(z)$ of multiplicity 0. We therefore need the KEY identity and its first derivative with respect to z:

$$\mathbf{L}(e^{zt}) = (z^2 + 2z - 3)e^{zt}\,,$$
$$\mathbf{L}(t\,e^{zt}) = (z^2 + 2z - 3)t\,e^{zt} + (2z+2)e^{zt}\,.$$

Evaluating these at $z = 1$ gives $\mathbf{L}(e^t) = 0$ and $\mathbf{L}(t\,e^t) = 4e^t$, whereby $\mathbf{L}(\frac{1}{4}t\,e^t) = e^t$. Evaluating these at $z = 0$ gives $\mathbf{L}(1) = -3$ and $\mathbf{L}(t) = -3t + 2$, which implies $\mathbf{L}(t + \frac{2}{3}) = -3t$. Hence, $Y_P(t) = \frac{1}{4}t\,e^t + t + \frac{2}{3}$. A general solution is

$$y = c_1 e^t + c_2 e^{-3t} + \tfrac{1}{4}t\,e^t + t + \tfrac{2}{3}\,.$$

Alternatively, to get the forcing term e^t you set $Y_P(t) = At\,e^t$, plug it into the equation, and solve for A; to get the forcing term $-3t$ you set $Y_P(t) = A_0 t + A_1$, plug it into the equation, and solve for A_0 and A_1. You get the same general solution.

(2) [5] The functions $1 + t$ and e^t are solutions of the equation

$$t\,\frac{d^2y}{dt^2} - (1+t)\frac{dy}{dt} + y = 0\,.$$

(You do not have to check that this is true.) Find a general solution of

$$t\,\frac{d^2y}{dt^2} - (1+t)\frac{dy}{dt} + y = t^2 e^t\,.$$

Solution. Divide the equation by t to bring it into normal form. A general solution of the associated homogeneous problem is

$$Y_H(t) = c_1(1+t) + c_2 e^t\,.$$

Use variation of parameters to find $Y_P(t)$ in the form

$$Y_P(t) = u_1(t)(1+t) + u_2(t)e^t\,,$$

where

$$u_1'(t)(1+t) + u_2'(t)e^t = 0\,,$$
$$u_1'(t)1 + u_2'(t)e^t = t\,e^t\,.$$

The solution of this linear algebraic system is $u_1'(t) = -e^t$ and $u_2'(t) = 1 + t$, whereby $u_1(t) = c_1 - e^t$ and $u_2(t) = c_2 + t + \frac{1}{2}t^2$. A general solution is

$$y = c_1(1+t) + c_2 e^t - e^t(1+t) + (t + \tfrac{1}{2}t^2)e^t\,.$$