**Quiz 8 Solutions, Math 246, Professor David Levermore**
**Wednesday, 16 April 2008**

(1) [4] Let $\mathbf{A} = \begin{pmatrix} 2 & -1+i3 \\ 1+i & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} i & 5 \\ -3 & -i \end{pmatrix}$.

(a) Compute $2\mathbf{A} - 3\mathbf{B}$.

**Solution:**

$$2\mathbf{A} - 3\mathbf{B} = \begin{pmatrix} 4 & -2+i6 \\ 2+i2 & 8 \end{pmatrix} + \begin{pmatrix} -i3 & -15 \\ 9 & i3 \end{pmatrix} = \begin{pmatrix} 4-i3 & -17+i6 \\ 11+i2 & 8+i3 \end{pmatrix}.$$

(b) Compute $\mathbf{AB}$.

**Solution:**

$$\mathbf{AB} = \begin{pmatrix} 2 & -1+i3 \\ 1+i & 4 \end{pmatrix} \begin{pmatrix} i & 5 \\ -3 & -i \end{pmatrix} = \begin{pmatrix} i2+3-i9 & 10+3+i \\ -1+i-12 & 5+i5-i4 \end{pmatrix} = \begin{pmatrix} 3-i7 & 13+i \\ -13+i & 5+i \end{pmatrix}.$$

(2) [3] Transform the equation $\dfrac{\mathrm{d}^3u}{\mathrm{d}t^3} + 2t\dfrac{\mathrm{d}^2u}{\mathrm{d}t^2} + 5u = \cos(3t)$ into a first-order system of ordinary differential equations.

**Solution:** Because the equation is third order, the first order system must have dimension three. The simplest such first order system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \cos(3t) - 5x_1 - 2tx_3 \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} u \\ u' \\ u'' \end{pmatrix}.$$

(3) [3] Consider two interconnected tanks filled with brine (salt water). The first tank contains 50 gallons and the second contains 100 gallons. Well stirred brine flows from the first tank to the second at a rate of 2 gallons per hour, and from the second to the first at the same rate. At $t = 0$ there are 100 oz of salt in the first tank and 50 oz in the second. Give an initial-value problem that governs the amount of salt in each tank as a function of time.

**Solution:** Let $S_1(t)$ be the amount of salt in the first tank and $S_2(t)$ be the amount of salt in the second tank. These are governed by the initial-value problem

$$\frac{\mathrm{d}S_1}{\mathrm{d}t} = \frac{S_2}{100}\,2 - \frac{S_1}{50}\,2\,, \qquad S_1(0) = 100\,,$$
$$\frac{\mathrm{d}S_2}{\mathrm{d}t} = \frac{S_1}{50}\,2 - \frac{S_2}{100}\,2\,, \qquad S_2(0) = 50\,.$$