Quiz 9 Solutions, Math 246, Professor David Levermore
Wednesday, 23 April 2008

(1) [2] Given that $e^{t\mathbf{A}} = \begin{pmatrix} \cos(4t) - \frac{3}{4}\sin(4t) & \frac{5}{4}\sin(4t) \\ -\frac{5}{4}\sin(4t) & \cos(4t) + \frac{3}{4}\sin(4t) \end{pmatrix}$,

solve the initial-value problem $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x}\,, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

Solution.

$$\begin{aligned}\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}(0) &= \begin{pmatrix} \cos(4t) - \frac{3}{4}\sin(4t) & \frac{5}{4}\sin(4t) \\ -\frac{5}{4}\sin(4t) & \cos(4t) + \frac{3}{4}\sin(4t) \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} 3\cos(4t) + 5\sin(4t) \\ 5\cos(4t) \end{pmatrix}.\end{aligned}$$

(2) [4] Let $\mathbf{A} = \begin{pmatrix} 5 & 4 \\ 4 & -1 \end{pmatrix}$. Compute $e^{t\mathbf{A}}$.

Solution. The characteristic polynomial is $p(z) = z^2 - 4z - 21 = (z-2)^2 - 25$. Hence,

$$\begin{aligned}e^{t\mathbf{A}} &= \mathbf{I}e^{2t}\cosh(5t) + (\mathbf{A} - 2\mathbf{I})e^{2t}\frac{\sinh(5t)}{5} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e^{2t}\cosh(5t) + \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} e^{2t}\frac{\sinh(5t)}{5} \\ &= e^{2t}\begin{pmatrix} \cosh(5t) + \frac{3}{5}\sinh(5t) & \frac{4}{5}\sinh(5t) \\ \frac{4}{5}\sinh(5t) & \cosh(5t) - \frac{3}{5}\sinh(5t) \end{pmatrix}.\end{aligned}$$

(3) [4] Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} t \\ 1 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$.

(a) Compute the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2](t)$.

Solution.

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \det\begin{pmatrix} t & t^2 \\ 1 & 2t \end{pmatrix} = 2t^2 - t^2 = t^2\,.$$

(b) Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x}$ wherever $W[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$.

Solution. Let $\mathbf{\Psi}(t) = \begin{pmatrix} t & t^2 \\ 1 & 2t \end{pmatrix}$. Because $\dfrac{\mathbf{\Psi}(t)}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{\Psi}(t)$, one has

$$\begin{aligned}\mathbf{A}(t) = \frac{\mathbf{\Psi}(t)}{\mathrm{d}t}\mathbf{\Psi}(t)^{-1} &= \begin{pmatrix} 1 & 2t \\ 0 & 2 \end{pmatrix}\begin{pmatrix} t & t^2 \\ 1 & 2t \end{pmatrix}^{-1} \\ &= \frac{1}{t^2}\begin{pmatrix} 1 & 2t \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2t & -t^2 \\ -1 & t \end{pmatrix} = \frac{1}{t^2}\begin{pmatrix} 0 & t^2 \\ -2 & 2t \end{pmatrix}.\end{aligned}$$