First In-Class Exam Solutions
Math 246, Fall 2008, Professor David Levermore

(1) [12] Suppose you are using numerical methods to approximate the solution of an initial-value problem over the time interval $[0, 10]$. By what factor would you expect the global error to decrease if you increase the number of time steps taken from 400 to 800 when you use the following explicit methods with a uniform time step h.

(a) Runge-Kutta method
Solution: This method is fourth order, so its error scales like h^4. When h decreases by a factor of 2 the error will therefore decrease by a factor of $2^4 = 16$.

(b) Heun-trapezoidal method
Solution: This method is second order, so its error scales like h^2. When h decreases by a factor of 2 the error will therefore decrease by a factor of $2^2 = 4$.

(c) Euler method
Solution: This method is first order, so its error scales like h. When h decreases by a factor of 2 the error will therefore decrease by a factor of 2.

(d) Heun-midpoint method
Solution: This method is second order, so its error scales like h^2. When h decreases by a factor of 2 the error will therefore decrease by a factor of $2^2 = 4$.

(2) [20] Find the explicit solution for each of the following initial-value problems and identify its interval of existence (interval of definition).

(a) $t\,\dfrac{\mathrm{d}w}{\mathrm{d}t} - 3w = t^2\,, \quad w(1) = 3.$

Solution: This equation is linear. Its linear normal form is

$$\frac{\mathrm{d}w}{\mathrm{d}t} - \frac{3}{t}\,w = t\,.$$

An integrating factor is $\exp\big(-\int_1^t \frac{3}{s}\,ds\big) = \exp(-3\log(t)) = t^{-3}$, so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(t^{-3}w\big) = t^{-3}\cdot t = t^{-2}\,, \qquad \Longrightarrow \qquad t^{-3}w = -t^{-1} + C\,.$$

The initial condition $w(1) = 3$ implies that $C = 1^{-3}\cdot 3 + 1^{-1} = 4$. Therefore

$$w = -t^2 + 4t^3\,, \quad \text{with interval of existence } t > 0\,.$$

(b) $\dfrac{\mathrm{d}z}{\mathrm{d}x} = \dfrac{6x^2}{3+z}\,, \quad z(0) = -1.$

Solution: This equation is separable. Its separated differential form is

$$(z+3)\,\mathrm{d}z = 6x^2\,\mathrm{d}x\,, \qquad \Longrightarrow \qquad \tfrac{1}{2}(z+3)^2 = 2x^3 + C\,.$$

The initial condition $z(0) = -1$ implies that $C = \frac{1}{2}(-1+3)^2 - 2\cdot 0^3 = 2$. Therefore $(z+3)^2 = 2(2x^2+2) = 4(x^3+1)$, which can be solved as

$$z = -3 + 2\sqrt{x^3+1}\,, \quad \text{with interval of existence } x > -1\,.$$

The positive square root is needed to satisfy the initial condition.

(3) [18] Consider the differential equation $\dfrac{\mathrm{d}x}{\mathrm{d}t} = (x+3)^2 x(4-x)$.

(a) Sketch its phase-line. Indicate all of the stationary (equilibrium) solutions and classify each as being either stable, unstable, or semistable.
(b) If $x(0) = 6$, how does the solution $x(t)$ behave as $t \to \infty$?
(c) If $x(0) = 2$, how does the solution $x(t)$ behave as $t \to \infty$?
(d) If $x(0) = -2$, how does the solution $x(t)$ behave as $t \to \infty$?
(e) If $x(0) = -6$, how does the solution $x(t)$ behave as $t \to \infty$?

Solution (a): The stationary solutions are $x = -3$, $x = 0$, and $x = 4$. A sign analysis of $(x+2)^3 x(4-x)$ shows that the phase-line for this equation is therefore

$$
\begin{array}{ccccccc}
- & & - & & + & & - \\
\leftarrow\leftarrow\leftarrow\leftarrow & \bullet & \leftarrow\leftarrow\leftarrow\leftarrow & \bullet & \rightarrow\rightarrow\rightarrow\rightarrow & \bullet & \leftarrow\leftarrow\leftarrow\leftarrow\ y \\
& -3 & & 0 & & 4 & \\
& \text{semistable} & & \text{unstable} & & \text{stable} &
\end{array}
$$

(b): The phase-line shows that if $x(0) = 6$ then $x(t) \to 4$ as $t \to \infty$.
(c): The phase-line shows that if $x(0) = 2$ then $x(t) \to 4$ as $t \to \infty$.
(d): The phase-line shows that if $x(0) = -2$ then $x(t) \to -3$ as $t \to \infty$.
(e): The phase-line shows that if $x(0) = -6$ then $x(t) \to -\infty$ as $t \to \infty$.

(4) [18] Consider the following MATLAB function M-file.

```
function [t,y] = solveit(ti, yi, tf, n)

h = (tf - ti)/n;
t = zeros(n + 1, 1);
y = zeros(n + 1, 1);
t(1) = ti;
y(1) = yi;
for k = 1:n
yhalf = y(k) + (h/2)*(3*y(k) + (y(k))^2);
t(k + 1) = t(k) + h;
y(k + 1) = y(k) + h*(3*yhalf + (yhalf)^2);
end
```

(a) What is the initial-value problem being approximated numerically?
(b) What is the numerical method being used?
(c) What are the output values of t(2) and y(2) that you would expect for input values of ti = 0, yi = 1, tf = 5, n = 25?

Solution (a): The initial-value problem being solved is $\dfrac{\mathrm{d}y}{\mathrm{d}t} = 3y + y^2, \quad y(\mathrm{ti})=\mathrm{yi}.$
(b): It is being approximated by the Heun-midpoint method.
(c): When ti = 0, yi = 1, tf = 5, n = 25 one has h = (tf - ti)/n = (5 - 0)/25 = .2, t(1) = ti = 0, and y(1) = yi = 1. Hence,
yhalf = y(1) + (h/2) (3 y(1) + y(1)^2) $= 1 + .1(3\cdot 1 + 1) = 1.4$,
t(2) = t(1) + h = 0 + .2 = .2 ,
y(2) = y(1) + h (3 yhalf - yhalf^2) $= 1 + .2(3\cdot 1.4 + (1.4)^2)$.

(5) [12] There are 240,000 mosquitoes in a certain area initially. In the absence of predators this population of mosquitoes would increase at a rate proportional to the current population and would double every three weeks. However, predators eat 100,000 mosquitoes per week at a constant rate.

(a) Write down an initial-value problem that governs the population of mosquitoes in the area at any positive time.

(b) Will this population increase or decrease over time?

Solution (a): Let $P(t)$ be the population of mosquitoes at time t weeks. Doubling every three weeks implies a growth rate of $\log(2)/3$. The initial-value problem satisfied by P is

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{\log(2)}{3} P - 100,000\,, \qquad P(0) = 240,000\,.$$

(b): Because

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{\log(2)}{3}\left(P - \frac{300,000}{\log(2)}\right),$$

the only stationary solution is $P = \frac{300{,}000}{\log(2)}$. A sign analysis of $\frac{\mathrm{d}P}{\mathrm{d}t}$ shows that the phase-line for this equation is therefore

$$\begin{array}{c} \qquad - \qquad\qquad\qquad + \\ \longleftarrow\longleftarrow\longleftarrow\longleftarrow\longleftarrow\longleftarrow \underset{\frac{300{,}000}{\log(2)}}{\bullet} \longrightarrow\longrightarrow\longrightarrow\longrightarrow\longrightarrow\longrightarrow P \end{array}$$

Because $\log(2) < 1$ implies $P(0) = 240,000 < \frac{300{,}000}{\log(2)}$, you see from the phase-line that the population $P(t)$ will decrease over time.

(6) [20] Give an implicit general solution to each of the following differential equations.

(a) $(xy^4 + 3y)\,\mathrm{d}x + (2x^2y^3 + 3x + e^y)\,\mathrm{d}y = 0\,.$

Solution: This differential form is *exact* because

$$\partial_y(xy^4 + 3y) = 4xy^3 + 3 \quad = \quad \partial_x(2x^2y^3 + 3x + e^y) = 4xy^3 + 3\,.$$

We can therefore find $H(x,y)$ such that

$$\partial_x H(x,y) = xy^4 + 3y\,, \qquad \partial_y H(x,y) = 2x^2y^3 + 3x + e^y\,.$$

Integrating the first equation with respect to x yields

$$H(x,y) = \int (xy^4 + 3y)\,\mathrm{d}x = \tfrac{1}{2}x^2y^4 + 3xy + h(y)\,.$$

Plugging this expression for $H(x,y)$ into the second equation gives

$$2x^2y^3 + 3x + h'(y) = \partial_y H(x,y) = 2x^2y^3 + 3x + e^y\,,$$

which yields $h'(y) = e^y$. Taking $h(y) = e^y$, the general solution is

$$\tfrac{1}{2}x^2y^4 + 3xy + e^y = C\,.$$

(b) $(3x^2y + 2xy + y^3)\,\mathrm{d}x + (x^2 + y^2)\,\mathrm{d}y = 0\,.$

Solution: This differential form is *not exact* because

$$\partial_y(3x^2y + 2xy + y^3) = 3x^2 + 2x + 3y^2 \quad \neq \quad \partial_x(x^2 + y^2) = 2x\,.$$

You therefore seek an *integrating factor* μ such that

$$\partial_y[(3x^2y + 2xy + y^3)\mu] = \partial_x[(x^2 + y^2)\mu]\,.$$

Expanding the derivatives yields

$$(3x^2y + 2xy + y^3)\partial_y\mu + (3x^2 + 2x + 3y^2)\mu = (x^2 + y^2)\partial_x\mu + 2x\mu\,.$$

If you set $\partial_y\mu = 0$ then this becomes

$$(3x^2 + 2x + 3y^2)\mu = (x^2 + y^2)\partial_x\mu + 2x\mu\,,$$

which reduces to

$$3(x^2 + y^2)\mu = (x^2 + y^2)\partial_x\mu\,.$$

This simplifies to $3\mu = \partial_x\mu$, whereby $\mu = e^{3x}$.

Because e^{3x} is an integrating factor, the differential form

$$(3x^2y + 2xy + y^3)e^{3x}\,\mathrm{d}x + (x^2 + y^2)e^{3x}\,\mathrm{d}y = 0 \quad \text{is exact}\,.$$

You can therefore find $H(x, y)$ such that

$$\partial_x H(x, y) = (3x^2y + 2xy + y^3)e^{3x}\,, \qquad \partial_y H(x, y) = (x^2 + y^2)e^{3x}\,.$$

Integrating the second equation with respect to y yields

$$\begin{aligned} H(x, y) &= \int (x^2 + y^2)e^{3x}\,\mathrm{d}y = \int x^2e^{3x} + y^2e^{3x}\,\mathrm{d}y \\ &= yx^2e^{3x} + \tfrac{1}{3}y^3e^{3x} + h(x)\,. \end{aligned}$$

Plugging this expression for $H(x, y)$ into the first equation gives

$$\begin{aligned} &2yxe^{3x} + 3yx^2e^{3x} + y^3e^{3x} + h'(x) \\ &\qquad = \partial_x H(x, y) = (3x^2y + 2xy + y^3)e^{3x}\,, \end{aligned}$$

which yields $h'(x) = 0$. Taking $h(x) = 0$, the general solution is

$$yx^2e^{3x} + \tfrac{1}{3}y^3e^{3x} = C\,.$$