Third In-Class Exam Solutions
Math 246, Fall 2008, Professor David Levermore

(1) [6] Consider the matrices

$$\mathbf{A} = \begin{pmatrix} 5 & 3 \\ 7 & 2 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 2 & -3 \\ -3 & 4 \end{pmatrix}.$$

Compute the matrices

(a) $\mathbf{AB}$ **Solution.** $\mathbf{AB} = \begin{pmatrix} 5 & 3 \\ 7 & 2 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 8 & -13 \end{pmatrix}$

(b) $\mathbf{B}^{-1}$ **Solution.** Because $\det(\mathbf{B}) = 2 \cdot 4 - (-3)(-3) = 8 - 9 = -1$,

$$\mathbf{B}^{-1} = \frac{1}{\det(\mathbf{B})}\begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} -4 & -3 \\ -3 & -2 \end{pmatrix}.$$

(2) [11] Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 4 & 4 \\ 4 & -2 \end{pmatrix}.$$

(a) Find all the eigenvalues of $\mathbf{A}$.

Solution. The characteristic polynomial of $\mathbf{A}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z - 24 = (z+4)(z-6).$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are -4 and 6.

(b) For each eigenvalue of $\mathbf{A}$ find an associated eigenvector.

Solution (using the Cayley-Hamilton method from notes). One has

$$\mathbf{A} + 4\mathbf{I} = \begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix}, \qquad \mathbf{A} - 6\mathbf{I} = \begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix}.$$

Every nonzero column of $\mathbf{A} - 6\mathbf{I}$ has the form

$$\alpha_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{for some } \alpha_1 \neq 0,$$

any of which is an eigenvector associated with -4. Similarly, every nonzero column of $\mathbf{A} + 4\mathbf{I}$ has the form

$$\alpha_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{for some } \alpha_2 \neq 0,$$

any of which is an eigenvector associated with 6.

(c) Compute $e^{t\mathbf{A}}$.

Solution. Because $p(z) = (z-1)^2 - 25 = (z-1)^2 - 5^2$,

$$
\begin{aligned}
e^{t\mathbf{A}} &= e^t \left[\mathbf{I}\cosh(5t) + (\mathbf{A}-\mathbf{I})\frac{\sinh(5t)}{5} \right] \\
&= e^t \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cosh(5t) + \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \frac{\sinh(5t)}{5} \right] \\
&= e^t \begin{pmatrix} \cosh(5t) + \frac{3}{5}\sinh(5t) & \frac{4}{5}\sinh(5t) \\ \frac{4}{5}\sinh(5t) & \cosh(5t) - \frac{3}{5}\sinh(5t) \end{pmatrix} .
\end{aligned}
$$

Alternative Solution. Because $\mathbf{A}$ has the eigenpairs

$$
\left(-4 \, , \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right) , \qquad \left(6 \, , \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right) ,
$$

you know that $\mathbf{A}$ is diagonalizable. Set

$$
\mathbf{V} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} , \qquad \mathbf{D} = \begin{pmatrix} -4 & 0 \\ 0 & 6 \end{pmatrix} .
$$

Then

$$
\begin{aligned}
e^{t\mathbf{A}} = \mathbf{V} e^{t\mathbf{D}} \mathbf{V}^{-1} &= \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} e^{-4t} & 0 \\ 0 & e^{6t} \end{pmatrix} \frac{1}{5} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \\
&= \frac{1}{5} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} e^{-4t} & -2e^{-4t} \\ 2e^{6t} & e^{6t} \end{pmatrix} \\
&= \frac{1}{5} \begin{pmatrix} e^{-4t} + 4e^{6t} & -2e^{-4t} + 2e^{6t} \\ -2e^{-4t} + 2e^{6t} & 4e^{-4t} + e^{6t} \end{pmatrix} .
\end{aligned}
$$

(3) [6] Transform the equation $\dfrac{\mathrm{d}^4 v}{\mathrm{d}t^4} + \cos(t)\dfrac{\mathrm{d}^3 v}{\mathrm{d}t^3} - 7v = \sin(t)$ into a first-order system of ordinary differential equations.

Solution: Because the equation is fourth order, the first order system must have dimension four. The simplest such first order system is

$$
\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ \sin(t) + 7x_1 - \cos(t)x_4 \end{pmatrix} , \qquad \text{where} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} v \\ v' \\ v'' \\ v''' \end{pmatrix} .
$$

(4) [10] Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ t^2 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t^2 \\ 2+t^4 \end{pmatrix}$.

(a) Compute the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2](t)$.

Solution.

$$
W[\mathbf{x}_1, \mathbf{x}_2](t) = \det \begin{pmatrix} 1 & t^2 \\ t^2 & 2+t^4 \end{pmatrix} = 2 + t^4 - t^4 = 2 \, .
$$

(b) Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to the system $\dfrac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x}$ wherever $W[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$.

Solution. Let $\boldsymbol{\Psi}(t) = \begin{pmatrix} 1 & t^2 \\ t^2 & 2+t^4 \end{pmatrix}$. Because $\dfrac{\boldsymbol{\Psi}(t)}{dt} = \mathbf{A}(t)\boldsymbol{\Psi}(t)$, one has

$$\mathbf{A}(t) = \frac{\boldsymbol{\Psi}(t)}{dt}\,\boldsymbol{\Psi}(t)^{-1} = \begin{pmatrix} 0 & 2t \\ 2t & 4t^3 \end{pmatrix} \begin{pmatrix} 1 & t^2 \\ t^2 & 2+t^4 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 0 & 2t \\ 2t & 4t^3 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 2+t^4 & -t^2 \\ -t^2 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -2t^3 & 2t \\ 4-2t^2 & 2t^3 \end{pmatrix} = \begin{pmatrix} -t^3 & t \\ 2-t^2 & t^3 \end{pmatrix}.$$

(c) Give a general solution to the system you found in part (b).

Solution. Because $W[\mathbf{x}_1, \mathbf{x}_2](t) = 2 \neq 0$, a general solution is

$$\mathbf{x} = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1 \begin{pmatrix} 1 \\ t^2 \end{pmatrix} + c_2 \begin{pmatrix} t^2 \\ 2+t^4 \end{pmatrix}.$$

(5) [5] Given that $e^{t\mathbf{A}} = \begin{pmatrix} \cosh(4t) + \frac{1}{2}\sinh(4t) & \frac{3}{4}\sinh(4t) \\ \sinh(4t) & \cosh(4t) - \frac{1}{2}\sinh(4t) \end{pmatrix}$, solve the initial-value problem

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Solution. The solution is given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{t\mathbf{A}} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} \cosh(4t) + \frac{1}{2}\sinh(4t) & \frac{3}{4}\sinh(4t) \\ \sinh(4t) & \cosh(4t) - \frac{1}{2}\sinh(4t) \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh(4t) + 2\sinh(4t) \\ 2\cosh(4t) \end{pmatrix}.$$

(6) [8] Consider two interconnected tanks filled with brine (salt water). The first tank contains 80 liters and the second contains 50 liters. Brine flows with a concentration of 2 grams of salt per liter flows into the first tank at a rate of 4 liters per hour. Well stirred brine flows from the first tank to the second at a rate of 7 liters per hour, from the second to the first at a rate of 3 liters per hour, and from the second into a drain at a rate of 4 liters per hour. At $t = 0$ there are 45 grams of salt in the first tank and 5 grams in the second. Give an initial-value problem that governs the amount of salt in each tank as a function of time.

Solution. The rates work out so there will always be 80 liters of brine in the first tank and 50 liters in the second. Let $S_1(t)$ and $S_2(t)$ be the grams of salt in the first and second tanks respectively. These are governed by the initial-value problem

$$\frac{dS_1}{dt} = 2\cdot 4 + \frac{S_2}{50}3 - \frac{S_1}{80}7, \qquad S_1(0) = 45,$$

$$\frac{dS_2}{dt} = \frac{S_1}{80}7 - \frac{S_2}{50}3 - \frac{S_2}{50}4, \qquad S_2(0) = 5.$$

(7) [12] Find a general solution for each of the following systems.

(a) $\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -5 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Solution. Let $\mathbf{A} = \begin{pmatrix} 2 & 4 \\ -5 & -2 \end{pmatrix}$. The characteristic polynomial of $\mathbf{A}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 16 = z^2 + 4^2\,,$$

which has roots $\pm i4$. Then, because $\mu = \frac{1}{2}\operatorname{tr}(\mathbf{A}) = 0$ and $\nu = 4$,

$$\begin{aligned} e^{t\mathbf{A}} &= \mathbf{I}\cos(4t) + \mathbf{A}\frac{\sin(4t)}{4} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(4t) + \begin{pmatrix} 2 & 4 \\ -5 & -2 \end{pmatrix}\frac{\sin(4t)}{4} \\ &= \begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) & \sin(4t) \\ -\frac{5}{4}\sin(4t) & \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix}. \end{aligned}$$

A general solution is therefore

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1\begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) \\ -\frac{5}{4}\sin(4t) \end{pmatrix} + c_2\begin{pmatrix} \sin(4t) \\ \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix}.$$

(b) $\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ -3 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Solution. Let $\mathbf{A} = \begin{pmatrix} 0 & 3 \\ -3 & -6 \end{pmatrix}$. The characteristic polynomial of $\mathbf{A}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - (-6)z + 9 = (z+3)^2\,,$$

which has the double root -3. Then, because $\mu = -3$ and $\nu = 0$,

$$\begin{aligned} e^{t\mathbf{A}} &= e^{-3t}\big[\mathbf{I} + (\mathbf{A} + 3\mathbf{I})t\big] = e^{-3t}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 3 \\ -3 & -3 \end{pmatrix}t\right] \\ &= e^{-3t}\begin{pmatrix} 1+3t & 3t \\ -3t & 1-3t \end{pmatrix}. \end{aligned}$$

A general solution is therefore

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1\begin{pmatrix} 1+3t \\ -3t \end{pmatrix} + c_2\begin{pmatrix} 3t \\ 1-3t \end{pmatrix}.$$

(8) [8] Sketch the phase portrait for each of the systems in the previous problem. Identify the type and stability of the origin.

Solution (a). The coefficient matrix $\mathbf{A}$ has the eigenvalues $\pm i4$. The origin is therefore a *center* and is thereby *stable*. Because $a_{21} = -5 < 0$, the phase portrait is a *clockwise center*.

Solution (b). The coefficient matrix $\mathbf{A}$ has the eigenvalue -3. Because

$$\mathbf{A} + 3\mathbf{I} = \begin{pmatrix} 3 & 3 \\ -3 & -3 \end{pmatrix},$$

it has the eigenpair

$$\left(-3\,,\begin{pmatrix}1\\-1\end{pmatrix}\right).$$

Because $\mathbf{A} \neq -3\mathbf{I}$, the origin is a *twist sink* (*improper nodal sink*) and is thereby *attracting* (*asymptotically stable*). Because $a_{21} = -5 < 0$, the phase portrait is a *clockwise twist sink*. There is one trajectory that moves towards the origin along each half of the line $y = -x$. Trajectories above the line $y = -x$ will approach the origin from below tangent to the line $y = -x$. Trajectories below the line $y = -x$ will approach the origin from above tangent to the line $y = -x$.

(9) [9] Consider the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}y\\-9x + x^3\end{pmatrix}.$$

(a) Find all of its stationary points.

Solution. The stationary points satisfy

$$0 = y\,, \qquad 0 = -9x + x^3 = x(x+3)(x-3)\,.$$

The second equation is satisfied if $x = 0$, $x = -3$, or $x = 3$. The stationary points of the system are therefore

$$(-3, 0)\,, \qquad (0, 0)\,, \qquad (3, 0)\,.$$

(b) Find a nonconstant function $H(x, y)$ such that every trajectory of the system satisfies $H(x, y) = c$ for some constant c.

Solution. Seek a solution of the first-order equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-9x + x^3}{y}\,.$$

This equation is separable. Integrate its separated form as

$$\int y\,\mathrm{d}y = \int -9x + x^3\,\mathrm{d}x\,,$$

which leads to $\frac{1}{2}y^2 = -\frac{9}{2}x^2 + \frac{1}{4}x^4 + c$. This has the form $H(x, y) = c$ with

$$H(x, y) = \tfrac{1}{2}y^2 + \tfrac{9}{2}x^2 - \tfrac{1}{4}x^4\,.$$

(10) [8] Compute the Laplace transform of $f(t) = u(t-5)\,e^{-2t}$ from its definition. (Here u is the unit step function.)

Solution. The definition of Laplace transform gives

$$\mathcal{L}[f](s) = \lim_{T\to\infty}\int_0^T e^{-st}u(t-5)\,e^{-2t}\,\mathrm{d}t = \lim_{T\to\infty}\int_5^T e^{-(s+2)t}\,\mathrm{d}t\,.$$

This limit diverges to $+\infty$ for $s \leq -2$ because in that case

$$\int_5^T e^{-(s+2)t}\,\mathrm{d}t \geq \int_5^T \mathrm{d}t = T - 5\,,$$

which clearly diverges to $+\infty$ as $T \to \infty$.

For $s > -2$ an integration shows that

$$\int_2^T e^{-(s+2)t}\,\mathrm{d}t = -\frac{e^{-(s+2)t}}{s+2}\bigg|_5^T = \frac{e^{-(s+2)5}}{s+2} - \frac{e^{-(s+2)T}}{s+2}\,.$$

Hence, for $s > -2$ one has that

$$\mathcal{L}[f](s) = \lim_{T\to\infty}\left(\frac{e^{-(s+2)5}}{s+2} - \frac{e^{-(s+2)T}}{s+2}\right) = \frac{e^{-(s+2)5}}{s+2} - \lim_{T\to\infty}\frac{e^{-(s+2)T}}{s+2} = \frac{e^{-5s-10}}{s+2}\,.$$

(11) [9] Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 4\frac{\mathrm{d}y}{\mathrm{d}t} + 20y = f(t)\,, \qquad y(0) = 3\,, \quad y'(0) = -6\,,$$

where

$$f(t) = \begin{cases} \frac{1}{2}t & \text{for } 0 \le t < 2\,, \\ e^{2-t} & \text{for } t \ge 2\,. \end{cases}$$

You may refer to the table on the last page. DO NOT take the inverse Laplace transform to find $y(t)$, just solve for $Y(s)$!

Solution. The Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) + 4\mathcal{L}[y'](s) + 20\mathcal{L}[y](s) = \mathcal{L}[f](s)\,,$$

where

$$\begin{aligned} \mathcal{L}[y](s) &= Y(s)\,, \\ \mathcal{L}[y'](s) &= sY(s) - y(0) = sY(s) - 3\,, \\ \mathcal{L}[y''](s) &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 3s + 6\,. \end{aligned}$$

The Laplace transform of the initial-value problem then becomes

$$\big(s^2Y(s) - 3s + 6\big) + 4\big(sY(s) - 3\big) + 20Y(s) = \mathcal{L}[f](s)\,,$$

which becomes

$$(s^2 + 4s + 20)Y(s) - 3s - 6 = \mathcal{L}[f](s)\,.$$

To compute $\mathcal{L}[f](s)$, first write f as

$$\begin{aligned} f(t) &= \big(u(t) - u(t-2)\big)\tfrac{1}{2}t + u(t-2)e^{2-t} \\ &= \tfrac{1}{2}t + u(t-2)\big(e^{2-t} - \tfrac{1}{2}t\big) \\ &= \tfrac{1}{2}t + u(t-2)\big(e^{-(t-2)} - \tfrac{1}{2}(t-2) - 1\big)\,. \end{aligned}$$

Referring to the table on the last page, item 1 with $n = 1$, item 6 with $c = 2$ and $f(t) = e^{-t} - \frac{1}{2}t - 1$, item 5 with $a = -1$ and $f(t) = 1$, item 1 with $n = 0$ and $n = 1$ then show that

$$\begin{aligned} \mathcal{L}[f](s) &= \mathcal{L}\big[\tfrac{1}{2}t\big](s) + \mathcal{L}\big[u(t-2)\big(e^{-(t-2)} - \tfrac{1}{2}(t-2) - 1\big)\big](s) \\ &= \mathcal{L}\big[\tfrac{1}{2}t\big](s) + e^{-2s}\mathcal{L}\big[e^{-t} - \tfrac{1}{2}t - 1\big](s) \\ &= \frac{1}{2s^2} + e^{-2s}\left(\frac{1}{s+1} - \frac{1}{2s^2} - \frac{1}{s}\right) = \frac{1 - e^{-2s}}{2s^2} + \frac{e^{-2s}}{s+1} - \frac{e^{-2s}}{s}\,. \end{aligned}$$

Hence, $Y(s)$ is given by

$$Y(s) = \frac{1}{s^2 + 4s + 20}\left(3s + 6 + \frac{1 - e^{-2s}}{2s^2} + \frac{e^{-2s}}{s+1} - \frac{e^{-2s}}{s}\right).$$

(12) [8] Find the inverse Laplace transforms of the function

$$F(s) = e^{-3s}\,\frac{s+2}{s^2 - 5s + 4}.$$

You may refer to the table on the last page.

Solution. The denominator factors as $s^2 - 5s + 4 = (s-1)(s-4)$. The partial fraction decomposition is

$$\frac{s+2}{s^2 - 5s + 4} = \frac{s+2}{(s-4)(s-1)} = \frac{2}{s-4} + \frac{-1}{s-1}.$$

The first item in the table with $n = 0$ and the fifth item with $f(t) = 1$ combine to show that

$$\mathcal{L}[e^{at}](s) = \frac{1}{s-a}.$$

In particular, setting $a = 4$ and $a = 1$ above we see that

$$\mathcal{L}[e^{4t}](s) = \frac{1}{s-4}, \qquad \mathcal{L}[e^{t}](s) = \frac{1}{s-1}.$$

Hence,

$$\frac{s+2}{s^2 - 5s + 4} = \frac{2}{s-4} - \frac{1}{s-1} = 2\mathcal{L}\big[e^{4t}\big](s) - \mathcal{L}\big[e^{t}\big](s) = \mathcal{L}\big[2e^{4t} - e^{t}\big](s).$$

By the sixth item in the table with $f(t) = 2e^{4t} - e^{t}$, we see that

$$\begin{aligned}\mathcal{L}\big[u(t-3)\big(2e^{4(t-3)} - e^{t-3}\big)\big](s) &= e^{-3s}\mathcal{L}\big[2e^{4t} - e^{t}\big](s) \\ &= e^{-3s}\,\frac{s+2}{s^2 - 5s + 4} = F(s).\end{aligned}$$

It follows that

$$\mathcal{L}^{-1}[F(s)](t) = 2u(t-3)e^{4t-12} - u(t-3)e^{t-3}.$$

A Short Table of Laplace Transforms

$\mathcal{L}[t^n](s) = \dfrac{n!}{s^{n+1}}$	for $s > 0$.
$\mathcal{L}[\cos(bt)](s) = \dfrac{s}{s^2 + b^2}$	for $s > 0$.
$\mathcal{L}[\sin(bt)](s) = \dfrac{b}{s^2 + b^2}$	for $s > 0$.
$\mathcal{L}[t^n f(t)](s) = (-1)^n F^{(n)}(s)$	where $F(s) = \mathcal{L}[f(t)](s)$.
$\mathcal{L}[e^{at} f(t)](s) = F(s-a)$	where $F(s) = \mathcal{L}[f(t)](s)$.
$\mathcal{L}[u(t-c)f(t-c)](s) = e^{-cs}F(s)$	where $F(s) = \mathcal{L}[f(t)](s)$ and u is the unit step function.