Solutions of Sample Problems for Second In-Class Exam
Math 246, Fall 2008, Professor David Levermore

(1) Give the interval of existence for the solution of the initial-value problem

$$\frac{\mathrm{d}^3x}{\mathrm{d}t^3} + \frac{\cos(3t)}{4-t}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{e^{-2t}}{1+t}, \qquad x(2) = x'(2) = x''(2) = 0\,.$$

Solution. The coefficient and forcing are both continuous over the interval $(-1, 4)$, which contains the initial time $t = 2$. The coefficient is not defined at $t = 4$ while the forcing is not defined at $t = -1$. The interval of existence is therefore $(-1, 4)$.

(2) Let L be a linear ordinary differential operator with constant coefficients. Suppose that all the roots of its characteristic polynomial (listed with their multiplicities) are $-2 + i3$, $-2 - i3$, $i7$, $i7$, $-i7$, $-i7$, 5, 5, 5, -3, 0, 0.

(a) Give the order of L.

Solution. There are 12 roots listed, so the degree of the characteristic polynomial is 12, whereby the order of L is 12.

(b) Give a general real solution of the homogeneous equation $\mathrm{L}y = 0$.

Solution. A general solution is

$$\begin{aligned} y = {} & c_1 e^{-2t}\cos(3t) + c_2 e^{-2t}\sin(3t) \\ & + c_3\cos(7t) + c_4\sin(7t) + c_5 t\,\cos(7t) + c_6 t\,\sin(7t) \\ & + c_7 e^{5t} + c_8 t\,e^{5t} + c_9 t^2 e^{5t} + c_{10} e^{-3t} + c_{11} + c_{12} t\,. \end{aligned}$$

The reasoning is as follows:
- the single conjugate pair $-2 \pm i3$ yields $e^{-2t}\cos(3t)$ and $e^{-2t}\sin(3t)$;
- the double conjugate pair $\pm i7$ yields

$$\cos(7t)\,, \quad \sin(7t)\,, \quad t\,\cos(7t)\,, \quad \text{and} \quad t\,\sin(7t)\,;$$

- the triple real root 5 yields e^{5t}, $t\,e^{5t}$, and $t^2 e^{5t}$;
- the single real root -3 yields e^{-3t};
- the double real root 0 yields 1 and t.

(3) Let $\mathrm{D} = \dfrac{\mathrm{d}}{\mathrm{d}t}$. Solve each of the following initial-value problems.

(a) $\mathrm{D}^2 y + 4\mathrm{D}y + 4y = 0\,, \qquad y(0) = 1\,, \quad y'(0) = 0\,.$

Solution. This is a constant coefficient, homogeneous, linear equation. Its characteristic polynomial is

$$p(z) = z^2 + 4z + 4 = (z+2)^2\,.$$

This has the double real root -2, which yields a general solution

$$y(t) = c_1 e^{-2t} + c_2 t\,e^{-2t}\,.$$

Because

$$y'(t) = -2c_1 e^{-2t} - 2c_2 t\,e^{-2t} + c_2 e^{-2t}\,,$$

when the initial conditions are imposed, one finds that

$$y(0) = c_1 = 1\,, \qquad y'(0) = -2c_1 + c_2 = 0\,.$$

These are solved to find $c_1 = 1$ and $c_2 = 2$. The solution of the initial-value problem is therefore

$$y(t) = e^{-2t} + 2t\,e^{-2t} = (1 + 2t)e^{-2t}\,.$$

(b) $\mathrm{D}^2 y + 9y = 20e^t\,, \qquad y(0) = 0\,, \quad y'(0) = 0\,.$

Solution. This is a constant coefficient, inhomogeneous, linear equation. Its characteristic polynomial is

$$p(z) = z^2 + 9 = z^2 + 3^2\,.$$

This has the conjugate pair of roots $\pm i3$, which yields a general solution of the associated homogeneous problem

$$y_H(t) = c_1 \cos(3t) + c_2 \sin(3t)\,.$$

The forcing $20e^t$ has degree $d = 0$ and characteristic $r + is = 1$, which is a root of $p(z)$ of multiplicity $m = 0$. A particular solution $y_P(t)$ can be found by the method of undetermined coefficients using either direct substitution or KEY identity evaluation.

Direct Substitution. Because $m = d = 0$, you seek a particular solution of the form

$$y_P(t) = Ae^t\,.$$

Because

$$y_P'(t) = Ae^t\,, \qquad y_P''(t) = Ae^t\,,$$

one sees that

$$\mathrm{L}y_P(t) = y_P''(t) + 9y_P(t) = Ae^t + 9Ae^t = 10Ae^t\,.$$

Setting $\mathrm{L}y_P(t) = 10Ae^t = 20e^t$, we see that $A = 2$. Hence, $y_P(t) = 2e^t$.

KEY Indentity Evaluations. Because $m + d = 0$, you only need to evaluate the KEY identity at $z = 1$, to find

$$\mathrm{L}(e^t) = p(1)e^t = (1^2 + 9)e^t = 10e^t\,.$$

Multiplying this equation by 2 yields $\mathrm{L}(2e^t) = 20e^t$. Hence, $y_P(t) = 2e^t$.

By either approach one finds $y_P(t) = 2e^t$, which yields the general solution

$$y(t) = c_1 \cos(3t) + c_2 \sin(3t) + 2e^t\,.$$

Because

$$y'(t) = -3c_1 \sin(3t) + 3c_2 \cos(3t) + 2e^t\,,$$

when the initial conditions are imposed, one finds that

$$y(0) = c_1 + 2 = 0\,, \qquad y'(0) = 3c_2 + 2 = 0\,.$$

These are solved to find $c_1 = -2$ and $c_2 = -\frac{2}{3}$. The solution of the initial-value problem is therefore

$$y(t) = -2\cos(3t) - \tfrac{2}{3}\sin(3t) + 2e^t\,.$$

(4) Let $\mathrm{D} = \dfrac{\mathrm{d}}{\mathrm{d}t}$. Give a general real solution for each of the following equations.

(a) $\mathrm{D}^2 y + 4\mathrm{D}y + 5y = 3\cos(2t)$.

Solution. This is a constant coefficient, inhomogeneous, linear equation. Its characteristic polynomial is

$$p(z) = z^2 + 4z + 5 = (z+2)^2 + 1 .$$

This has the conjugate pair of roots $-2 \pm i$, which yields a general solution of the associated homogeneous problem

$$y_H(t) = c_1 e^{-2t}\cos(t) + c_2 e^{-2t}\sin(t) .$$

The forcing $3\cos(2t)$ has degree $d = 0$ and characteristic $r + is = i2$, which is a root of $p(z)$ of multiplicity $m = 0$. A particular solution $y_P(t)$ can be found by the method of undetermined coefficients using either direct substitution or KEY identity evaluation.

Direct Substitution. Because $m = d = 0$, you seek a particular solution of the form

$$y_P(t) = A\cos(2t) + B\sin(2t) .$$

Because

$$\begin{aligned} y_P'(t) &= -2A\sin(2t) + 2B\cos(2t) , \\ y_P''(t) &= -4A\cos(2t) - 4B\sin(2t) , \end{aligned}$$

one sees that

$$\begin{aligned} \mathrm{L}y_P(t) &= y_P''(t) + 4y_P'(t) + 5y_P(t) \\ &= \big[-4A\cos(2t) - 4B\sin(2t)\big] + 4\big[-2A\sin(2t) + 2B\cos(2t)\big] \\ &\quad + 5\big[A\cos(2t) + B\sin(2t)\big] \\ &= (A + 8B)\cos(2t) + (B - 8A)\sin(2t) . \end{aligned}$$

Setting $\mathrm{L}y_P(t) = (A + 8B)\cos(2t) + (B - 8A)\sin(2t) = 3\cos(2t)$, we see that

$$A + 8B = 3 , \qquad B - 8A = 0 .$$

One finds that $A = \frac{3}{65}$ and $B = \frac{24}{65}$. Hence, $y_P(t) = \frac{3}{65}\cos(2t) + \frac{24}{65}\sin(2t)$. A general solution is therefore

$$y = c_1 e^{-2t}\cos(t) + c_2 e^{-2t}\sin(t) + \tfrac{3}{65}\cos(2t) + \tfrac{24}{65}\sin(2t) .$$

KEY Indentity Evaluations. Because $m + d = 0$, you only need to evaluate the KEY identity at $z = i2$, to find

$$\mathrm{L}\big(e^{i2t}\big) = p(i2)e^{i2t} = \big((i2)^2 + 4(i2) + 5\big)e^{i2t} = (1 + i8)e^{i2t} .$$

Because the forcing $3\cos(2t) = 3\,\mathrm{Re}\big(e^{i2t}\big)$, you divide the above by $1 + i8$ and multiply by 3 to find

$$\mathrm{L}\left(\frac{3}{1 + i8}\, e^{i2t}\right) = 3e^{i2t} .$$

Hence,

$$\begin{aligned} y_P(t) &= \mathrm{Re}\bigg(\frac{3}{1+i8}\, e^{i2t} \bigg) = \mathrm{Re}\bigg(\frac{3(1-i8)}{1^2+8^2}\, e^{i2t} \bigg) = \tfrac{3}{65}\,\mathrm{Re}\big((1-i8)e^{i2t}\big) \\ &= \tfrac{3}{65}\big(\cos(2t) + 8\sin(2t) \big) = \tfrac{3}{65}\cos(2t) + \tfrac{24}{65}\sin(2t)\,. \end{aligned}$$

A general solution is therefore

$$y = c_1 e^{-2t}\cos(t) + c_2 e^{-2t}\sin(t) + \tfrac{3}{65}\cos(2t) + \tfrac{24}{65}\sin(2t)\,.$$

(b) $\mathrm{D}^2 y - y = t\, e^t$.

Solution. This is a constant coefficient, inhomogeneous, linear equation. Its characteristic polynomial is

$$p(z) = z^2 - 1 = (z+1)(z-1)\,.$$

This has the real roots -1 and 1, which yields a general solution of the associated homogeneous problem

$$y_H(t) = c_1 e^{-t} + c_2 e^t\,.$$

The forcing $t\, e^t$ has degree $d = 1$ and characteristic $r + is = 1$, which is a root of $p(z)$ of multiplicity $m = 1$. A particular solution $y_P(t)$ can be found by the method of undetermined coefficients using either direct substitution or KEY identity evaluation.

Direct Substitution. Because $m = 1$ and $d = 1$, you seek a particular solution of the form

$$y_P(t) = \big(A_0 t^2 + A_1 t\big)\, e^t\,,$$

Because

$$\begin{aligned} y_P'(t) &= \big(A_0 t^2 + A_1 t\big)\, e^t + \big(2A_0 t + A_1\big)\, e^t \\ &= \big(A_0 t^2 + (2A_0 + A_1)t + A_1\big)\, e^t\,, \\ y_P''(t) &= \big(A_0 t^2 + (2A_0 + A_1)t + A_1\big)\, e^t + \big(2A_0 t + (2A_0 + A_1)\big)\, e^t \\ &= \big(A_0 t^2 + (4A_0 + A_1)t + 2A_0 + 2A_1\big)\, e^t\,, \end{aligned}$$

one sees that

$$\begin{aligned} \mathrm{L}y_P(t) &= y_P''(t) - y_P(t) \\ &= \big(A_0 t^2 + (4A_0 + A_1)t + 2A_0 + 2A_1\big)\, e^t - \big(A_0 t^2 + A_1 t\big)\, e^t \\ &= \big(4A_0 t + 2A_0 + 2A_1\big)\, e^t = 4A_0 t\, e^t + 2(A_0 + A_1)e^t\,. \end{aligned}$$

Setting $\mathrm{L}y_P(t) = 4A_0 t\, e^t + 2(A_0 + A_1)e^t = t\, e^t$, we obtain $4A_0 = 1$ and $A_0 + A_1 = 0$. It follows that $A_0 = \frac{1}{4}$ and $A_1 = -\frac{1}{4}$. Hence, $y_P(t) = \frac{1}{4}(t^2 - t)\, e^t$. A general solution is therefore

$$y = c_1 e^{-t} + c_2 e^t + \tfrac{1}{4}(t^2 - t)\, e^t\,.$$

KEY Indentity Evaluations. Because $m+d=2$, you need the KEY identity and its first two derivatives

$$\begin{aligned} \mathrm{L}\big(e^{zt}\big) &= (z^2-1)e^{zt}\,, \\ \mathrm{L}\big(t\,e^{zt}\big) &= (z^2-1)t\,e^{zt} + 2ze^{zt}\,, \\ \mathrm{L}\big(t^2e^{zt}\big) &= (z^2-1)t^2e^{zt} + 4zt\,e^{zt} + 2e^{zt}\,. \end{aligned}$$

Evaluate these at $z=1$ to find

$$\mathrm{L}\big(e^t\big) = 0\,, \qquad \mathrm{L}\big(t\,e^t\big) = 2e^t\,, \qquad \mathrm{L}\big(t^2e^t\big) = 4t\,e^t + 2e^t\,.$$

Subtracting the second equation from the third yields

$$\mathrm{L}\big(t^2e^t - t\,e^t\big) = 4t\,e^t\,.$$

Dividing this equation by t gives $\mathrm{L}\big(\frac{1}{4}(t^2-t)\,e^t\big) = t\,e^t$. Hence, $y_P(t) = \frac{1}{4}(t^2-t)\,e^t$. A general solution is therefore

$$y = c_1e^{-t} + c_2e^t + \tfrac{1}{4}(t^2-t)\,e^t\,.$$

(c) $\mathrm{D}^2y - y = \dfrac{1}{1+e^t}\,.$

Solution. This is a constant coefficient, inhomogeneous, linear equation. Its characteristic polynomial is

$$p(z) = z^2 - 1 = (z-1)(z+1)\,.$$

This has the real roots 1 and -1, which yields a general solution of the associated homogeneous problem

$$y_H(t) = c_1e^t + c_2e^{-t}\,.$$

The forcing does not have the form needed for undertermined coefficients. You must therefore use either the Green function method or the variation of parameters method.

Green Function. The Green function $g(t)$ satisfies

$$\mathrm{D}^2g - g = 0\,, \qquad g(0) = 0\,, \quad g'(0) = 1\,.$$

Set $g(t) = c_1e^t + c_2e^{-t}$. The first initial condition implies $g(0) = c_1 + c_2 = 0$. Because $g'(t) = c_1e^t - c_2e^{-t}$, the second initial condition yields $g'(0) = c_1 - c_2 = 1$. It follows that $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}$, whereby $g(t) = \frac{1}{2}(e^t - e^{-t})$. Hence, a particular solution is

$$\begin{aligned} Y_P(t) &= \tfrac{1}{2}\int_0^t \big(e^{t-s} - e^{-t+s}\big)\frac{1}{1+e^s}\,\mathrm{d}s \\ &= \tfrac{1}{2}e^t\int_0^t \frac{e^{-s}}{1+e^s}\,\mathrm{d}s - \tfrac{1}{2}e^{-t}\int_0^t \frac{e^s}{1+e^s}\,\mathrm{d}s\,. \end{aligned}$$

The definite integrals on the right-hand side can be evaluated as

$$\int_0^t \frac{e^{-s}}{1+e^s}\,\mathrm{d}s = \int_0^t \frac{e^{-2s}}{e^{-s}+1}\,\mathrm{d}s = \int_0^t e^{-s} - \frac{e^{-s}}{e^{-s}+1}\,\mathrm{d}s$$

$$= \Big[-e^{-s} + \log\big(e^{-s}+1\big)\Big]\Big|_{s=0}^{t} = 1 - e^{-t} + \log\left(\frac{e^{-t}+1}{2}\right),$$

$$\int_0^t \frac{e^s}{1+e^s}\,\mathrm{d}s = \log\big(1+e^s\big)\Big|_{s=0}^{t} = \log\left(\frac{1+e^t}{2}\right).$$

Hence, the particular solution $Y_P(t)$ is given by

$$Y_P(t) = \tfrac{1}{2}\left[e^t - 1 + e^t\log\left(\frac{e^{-t}+1}{2}\right)\right] - \tfrac{1}{2}e^{-t}\log\left(\frac{1+e^t}{2}\right).$$

A general solution is therefore $y = Y_H(t) + Y_P(t)$ where $Y_H(t)$ and $Y_P(t)$ are given above.

Variation of Parameters. Seek a solution in the form

$$y = u_1(t)\,e^t + u_2(t)\,e^{-t},$$

where $u_1(t)$ and $u_2(t)$ satisfy

$$u_1'(t)\,e^t + u_2'(t)\,e^{-t} = 0,$$

$$u_1'(t)\,e^t - u_2'(t)\,e^{-t} = \frac{1}{1+e^t}.$$

Solve this system to obtain

$$u_1'(t) = \tfrac{1}{2}\frac{e^{-t}}{1+e^t}, \qquad u_2'(t) = -\tfrac{1}{2}\frac{e^t}{1+e^t}.$$

Integrate these equations to find

$$u_1(t) = \tfrac{1}{2}\int \frac{e^{-t}}{1+e^t}\,\mathrm{d}t = \tfrac{1}{2}\int \frac{e^{-2t}}{e^{-t}+1}\,\mathrm{d}t$$

$$= \tfrac{1}{2}\int e^{-t} - \frac{e^{-t}}{e^{-t}+1}\,\mathrm{d}t = -\tfrac{1}{2}e^{-t} + \tfrac{1}{2}\log\big(e^{-t}+1\big) + c_1,$$

$$u_2(t) = -\tfrac{1}{2}\int \frac{e^t}{1+e^t}\,\mathrm{d}t = -\tfrac{1}{2}\log\big(1+e^t\big) + c_2.$$

A general solution is therefore

$$y = c_1e^t + c_2e^{-t} - \tfrac{1}{2} + \tfrac{1}{2}e^t\log\big(e^{-t}+1\big) - \tfrac{1}{2}e^{-t}\log\big(1+e^t\big).$$

(5) Let $\mathrm{D} = \dfrac{\mathrm{d}}{\mathrm{d}t}$. Consider the equation

$$\mathrm{L}y = \mathrm{D}^2 y - 6\mathrm{D}y + 25y = e^{t^2} .$$

(a) Compute the Green function $g(t)$ associated with L.

Solution. The Green function $g(t)$ satisfies

$$\mathrm{D}^2 g - 6\mathrm{D}g + 25g = 0 , \qquad g(0) = 0 , \quad g'(0) = 1 .$$

The characteristic polynomial of L is $p(z) = z^2 - 6z + 25 = (z-3)^2 + 4^2$, which has roots $3 \pm i4$. Set $g(t) = c_1 e^{3t}\cos(4t) + c_2 e^{3t}\sin(4t)$. The first initial condition implies $g(0) = c_1 = 0$, whereby $g(t) = c_2 e^{3t}\sin(4t)$. Because $g'(t) = 3c_2 e^{3t}\sin(4t) + 4c_2 e^{3t}\cos(4t)$, the second initial condition implies $g'(0) = 4c_2 = 1$, whereby $c_2 = \frac{1}{4}$. The Green function associated with L is therefore given by

$$g(t) = \tfrac{1}{4} e^{3t} \sin(4t) .$$

(b) Use the Green function to express a particular solution $Y_P(t)$ in terms of definite integrals.

Solution. A particular solution $Y_P(t)$ is given by

$$Y_P(t) = \int_0^t g(t-s) e^{s^2}\, \mathrm{d}s = \tfrac{1}{4} \int_0^t e^{3(t-s)} \sin\bigl(4(t-s)\bigr) e^{s^2}\, \mathrm{d}s .$$

Because $\sin\bigl(4(t-s)\bigr) = \sin(4t)\cos(4s) - \cos(4t)\sin(4t)$, this particular solution is given in terms of definite integrals as

$$Y_P(t) = \tfrac{1}{4} e^{3t} \sin(4t) \int_0^t e^{-3s} \cos(4s) e^{s^2}\, \mathrm{d}s - \tfrac{1}{4} e^{3t} \cos(4t) \int_0^t e^{-3s} \sin(4s) e^{s^2}\, \mathrm{d}s .$$

Remark: The above definite integrals cannot be evaluated analytically.

(6) The functions x and x^2 are solutions of the homogeneous equation

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2x \frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0 \qquad \text{over } x > 0 .$$

(You do not have to check that this is true!)

(a) Compute their Wronskian.

Solution. The Wronskian is

$$W[x, x^2](x) = \det \begin{pmatrix} x & x^2 \\ 1 & 2x \end{pmatrix} = x(2x) - 1x^2 = 2x^2 - x^2 = x^2 .$$

(b) Give a general solution of the equation

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2x \frac{\mathrm{d}y}{\mathrm{d}x} + 2y = x^3 e^x \qquad \text{over } x > 0 .$$

You may express the solution in terms of definite integrals.

Solution. Because $W[x, x^2](x) = x^2 \neq 0$ over $x > 0$, the functions x and x^2 are linearly independent. A general solution of the associated homogeneous problem is

$$y_H(x) = c_1 x + c_2 x^2 .$$

Because this problem does not have constant coefficients, you must use the method of variation of parameters to find a particular solution $y_P(x)$. First, divide by x^2 to bring the equation into its normal form

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \frac{2}{x}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x^2}\, y = xe^x \qquad \text{over } x > 0 .$$

Seek a solution in the form

$$y = u_1(x)x + u_2(x)x^2 ,$$

where $u_1'(x)$ and $u_2'(x)$ satisfy

$$\begin{aligned} u_1'(x)x + u_2'(x)x^2 &= 0 , \\ u_1'(x)1 + u_2'(x)2x &= x\, e^x . \end{aligned}$$

Solve this system to obtain

$$u_1'(x) = -x\, e^x , \qquad u_2'(x) = e^x .$$

Integrate these equations to find

$$u_1(x) = c_1 + (1 - x)e^x , \qquad u_2(x) = c_2 + e^x .$$

A general solution is therefore

$$y = c_1 x + c_2 x^2 + (1 - x)e^x x + e^x x^2 = c_1 x + c_2 x^2 + x\, e^x .$$

(7) What answer will be produced by the following MATLAB commands?

```
>> ode1 = 'D2y + 2*Dy + 5*y = 16*exp(t)';
>> dsolve(ode1, 't')
ans =
```

Solution. The commands ask MATLAB to give the general solution of the equation

$$\mathrm{D}^2 y + 2\mathrm{D}y + 5y = 16e^t , \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t} .$$

MATLAB will produce the answer

```
2*exp(t) + C1*exp(–t)*sin(2*t) + C2*exp(–t)*cos(2*t)
```

This can be seen as follows. This is a constant coefficient, inhomogeneous, linear equation. The characteristic polynomial is

$$p(z) = z^2 + 2z + 5 = (z + 1)^2 + 4 = (z + 1)^2 + 2^2 .$$

Its roots are the conjugate pair $-1 \pm i2$. A general solution of the associated homogeneous problem is

$$y_H(t) = c_1 e^{-t} \cos(2t) + c_2 e^{-t} \sin(2t) .$$

The forcing $16e^t$ has degree $d = 0$ and characteristic $r + is = 1$, which is a root of $p(z)$ of multiplicity $m = 0$. A particular solution $y_P(t)$ can be found by the method of undetermined coefficients using either direct substitution or KEY identity evaluation.

Direct Substitution. Because $m = d = 0$, you seek a particular solution of the form

$$y_P(t) = Ae^t .$$

Because

$$y_P'(t) = Ae^t , \qquad y_P''(t) = Ae^t ,$$

one sees that

$$\mathrm{L}y_P(t) = y_P''(t) + 2y_P'(t) + 5y_P(t) = [Ae^t] + 2[Ae^t] + 5[Ae^t] = 8Ae^t .$$

Setting $\mathrm{L}y_P(t) = 8Ae^t = 16e^t$, we see that $A = 2$. Hence, $y_P(t) = 2e^t$.

KEY Indentity Evaluations. Because $m + d = 0$, you only need to evaluate the KEY identity at $z = 1$, to find

$$\mathrm{L}(e^t) = p(1)e^t = (1^2 + 2 \cdot 1 + 5)e^t = 8e^t .$$

Multiply this by 2 to obtain $\mathrm{L}(2e^t) = 16e^t$. Hence, $y_P(t) = 2e^t$.

By either approach you find $y_P(t) = 2e^t$. A general solution is therefore

$$y = c_1 e^{-t} \cos(2t) + c_2 e^{-t} \sin(2t) + 2e^t .$$

Up to notational differnces, this is the answer that MATLAB produces.

(8) The vertical displacement of a mass on a spring is given by

$$z(t) = \sqrt{3}\cos(2t) + \sin(2t) .$$

Express this in the form $z(t) = A\cos(\omega t - \delta)$, identifying the amplitude and phase of the oscillation.

Solution. The displacement takes the form

$$z(t) = 2\cos(2t - \tfrac{\pi}{6}) ,$$

where the amplitude is 2 and the phase is $\frac{\pi}{6}$. There are several approaches to the problem. Here are two.

One approach that requires no memorization other than the addition formula for cosine is the following. Because

$$A\cos(\omega t - \delta) = A\cos(\delta)\cos(\omega t) + A\sin(\delta)\sin(\omega t) ,$$

this form will be equal to $z(t)$ provided $\omega = 2$ and

$$A\cos(\delta) = \sqrt{3} , \qquad A\sin(\delta) = 1 .$$

In other words, (A, δ) are the polar coordinates (radius, angle) of the point in the plane with Cartesian coordinates $(\sqrt{3}, 1)$. One finds that the amplitude A is given by

$$A = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = \sqrt{3 + 1} = \sqrt{4} = 2 ,$$

while the phase δ is in the first quadrant and is given by (for example)

$$\delta = \sin^{-1}\left(\frac{1}{A}\right) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6} .$$

Another approach requires you to memorize special formulas for both the amplitude and phase of functions of the form

$$c_1 \cos(\omega t) + c_2 \sin(\omega t)\,.$$

The formula for the amplitude is easy to remember because c_1 and c_2 appear in it symmetrically. It gives

$$A = \sqrt{c_1^2 + c_2^2} = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2\,.$$

Formulas for phase are trickier because c_1 and c_2 do not play symmetric roles. Because both c_1 and c_2 are positive, δ is in the first quadrant and is given by

$$\delta = \cos^{-1}\left(\frac{c_2}{A}\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}\,.$$

This formula holds whenever $c_1 \geq 0$, otherwise its result must be subtracted from 2π. The most common mistake made by those who choose this approach is to confuse the roles of c_1 and c_2. One way to keep these roles straight is to remember this formula verbally as

$$\text{phase} = \begin{cases} \cos^{-1}\left(\dfrac{\text{coefficient of cosine}}{\text{amplitude}}\right) & \text{for coefficient of sine nonnegative}\,, \\ 2\pi - \cos^{-1}\left(\dfrac{\text{coefficient of cosine}}{\text{amplitude}}\right) & \text{for coefficient of sine negative}\,. \end{cases}$$

(9) When a mass of 4 grams is hung vertically from a spring, at rest it stretches the spring 9.8 cm. (Gravitational acceleration is $g = 980$ cm/sec^2.) At $t = 0$ the mass is displaced 3 cm above its equilibrium position and is released with no initial velocity. It moves in a medium that imparts a drag force of 2 dynes (1 dyne = 1 gram cm/sec^2) when the speed of the mass is 4 cm/sec. There are no other forces. (Assume that the spring force is proportional to displacement and that the drag force is proportional to velocity.)

(a) Formulate an initial-value problem that governs the motion of the mass for $t > 0$. (DO NOT solve this initial-value problem, just write it down!)

Solution. Let $h(t)$ be the displacement of the mass from its equilibrium (rest) position at time t in centimeters, with upward displacements being positive. The governing initial-value problem then has the form

$$m\frac{\mathrm{d}^2h}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}h}{\mathrm{d}t} + kh = 0\,, \qquad h(0) = 3\,, \quad h'(0) = 0\,,$$

where m is the mass, γ is the drag coefficient, and k is the spring constant. The problem says that $m = 4$ grams. The spring constant is obtained by balancing the weight of the mass ($mg = 4 \cdot 980$ dynes) with the force applied by the spring when it is stetched 9.8 cm. This gives $k\,9.8 = 4 \cdot 980$, or

$$k = \frac{4 \cdot 980}{9.8} = 400 \quad \text{dynes/cm}\,.$$

The drag coefficient is obtained by balanceing the force of 2 dynes with the drag force imparted by the medium when the speed of the mass is 4 cm/sec. This gives $\gamma\,4 = 2$, or

$$\gamma = \frac{2}{4} = \frac{1}{2} \quad \text{dynes sec/cm}\,.$$

The governing initial-value problem is therefore

$$4\frac{\mathrm{d}^2 h}{\mathrm{d}t^2} + \frac{1}{2}\,\frac{\mathrm{d}h}{\mathrm{d}t} + 400h = 0\,, \qquad h(0) = 3\,, \quad h'(0) = 0\,,$$

If you had chosen downward displacements to be positive then the governing initial-value problem would be the same except for the first initial condition, which would then be $h(0) = -3$.

(b) What is the natural frequency of the spring?

Solution. The natural frequency of the spring is given by

$$\omega_o = \sqrt{\frac{k}{m}} = \sqrt{\frac{4 \cdot 980}{4 \cdot 9.8}} = \sqrt{100} = 10 \quad 1/\text{sec}\,.$$

(c) Show that the system is under damped and find its quasifrequency.

Solution. The characteristic polynomial is

$$p(z) = z^2 + \frac{1}{8}z + 100 = \left(z + \frac{1}{16}\right)^2 + 100 - \frac{1}{16^2}\,,$$

which has a conjugate pair of roots. The system is therefore under damped. The roots are $-\frac{1}{16} \pm i\nu$ where

$$\nu = \sqrt{100 - \frac{1}{16^2}} \quad 1/\text{sec}\,.$$

This is the quasifrequency.