Sample Problems for Third In-Class Exam
Math 246, Spring 2008, Professor David Levermore

(1) Compute the Laplace transform of $f(t) = t\,e^{3t}$ from its definition.

(2) Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 4\frac{\mathrm{d}y}{\mathrm{d}t} + 13y = f(t)\,, \qquad y(0) = 4\,, \quad y'(0) = 1\,,$$

where

$$f(t) = \begin{cases} \cos(t) & \text{for } 0 \le t < 2\pi\,, \\ t - 2\pi & \text{for } t \ge 2\pi\,. \end{cases}$$

You may refer to the table on the last page. DO NOT take the inverse Laplace transform to find $y(t)$, just solve for $Y(s)$!

(3) Find the inverse Laplace transforms of the following functions. You may refer to the table on the last page.

(a) $F(s) = \dfrac{2}{(s+5)^2}$,

(b) $F(s) = \dfrac{3s}{s^2 - s - 6}$,

(c) $F(s) = \dfrac{(s-2)e^{-3s}}{s^2 - 4s + 5}$.

(4) Consider the matrices

$$\mathbf{A} = \begin{pmatrix} -i2 & 1+i \\ 2+i & -4 \end{pmatrix}\,, \qquad \mathbf{B} = \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix}\,.$$

Compute the matrices
(a) $\mathbf{A}^T$,
(b) $\overline{\mathbf{A}}$,
(c) $\mathbf{A}^*$,
(d) $5\mathbf{A} - \mathbf{B}$,
(e) $\mathbf{AB}$,
(f) $\mathbf{B}^{-1}$.

(5) Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 3 \\ 4 & -1 \end{pmatrix}\,.$$

(a) Find all the eigenvalues of $\mathbf{A}$.
(b) For each eigenvalue of $\mathbf{A}$ find all of its eigenvectors.
(c) Diagonalize $\mathbf{A}$.

(6) Given that 1 is an eigenvalue of the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & -1 & 3 \end{pmatrix},$$

find all the eigenvectors of $\mathbf{A}$ associated with 1.

(7) Transform the equation $\dfrac{\mathrm{d}^3u}{\mathrm{d}t^3} + t^2\dfrac{\mathrm{d}u}{\mathrm{d}t} - 3u = \sinh(2t)$ into a first-order system of ordinary differential equations.

(8) Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} t^2+3 \\ 2t \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t^3 \\ 3t^2 \end{pmatrix}$.

(a) Compute the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2](t)$.

(b) Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x}$ wherever $W[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$.

(9) Consider two interconnected tanks filled with brine (salt water). The first tank contains 100 liters and the second contains 50 liters. Brine flows with a concentration of 2 grams of salt per liter flows into the first tank at a rate of 3 liters per hour. Well stirred brine flows from the first tank to the second at a rate of 5 liters per hour, from the second to the first at a rate of 2 liters per hour, and from the second into a drain at a rate of 3 liters per hour. At $t = 0$ there are 5 grams of salt in the first tank and 20 grams in the second. Give an initial-value problem that governs the amount of salt in each tank as a function of time.

(10) Solve each of the following initial-value problems.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$

(11) Find a general solution for each of the following systems.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

(12) Sketch a phase portrait for each of the systems in Problem 5. Indicate some typical trajectories.

(13) Compute $e^{t\mathbf{A}}$ for $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$.

(14) Consider the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y+1 \\ 4x - x^2 \end{pmatrix}.$$

(a) Find all of its stationary points.
(b) Find a nonconstant function $H(x, y)$ such that every trajectory of the system satisfies $H(x, y) = c$ for some constant c.
(c) Sketch a phase portrait of the system. Indicate its stationary points and some typical trajectories.
(d) Identify each stationary point as being either stable or unstable.

A Short Table of Laplace Transforms

$$\mathcal{L}[t^n](s) = \frac{n!}{s^{n+1}} \qquad \text{for } s > 0\,.$$

$$\mathcal{L}[\cos(bt)](s) = \frac{s}{s^2 + b^2} \qquad \text{for } s > 0\,.$$

$$\mathcal{L}[\sin(bt)](s) = \frac{b}{s^2 + b^2} \qquad \text{for } s > 0\,.$$

$$\mathcal{L}[t^n f(t)](s) = (-1)^n F^{(n)}(s) \qquad \text{where } F(s) = \mathcal{L}[f(t)](s)\,.$$

$$\mathcal{L}[e^{at} f(t)](s) = F(s-a) \qquad \text{where } F(s) = \mathcal{L}[f(t)](s)\,.$$

$$\mathcal{L}[u(t-c)f(t-c)](s) = e^{-cs}F(s) \qquad \text{where } F(s) = \mathcal{L}[f(t)](s)$$

and u is the unit step function.