Sample Final Exam Problems, Math 246, Fall 2008

(1) Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = (16 - y^2)^2 y$ over the interval $-5 \le y \le 5$.
   (a) Identify its equilibrium (stationary) points and classify their stability.
   (b) Sketch its phase-line portrait in this interval.
   (c) If $y(0) = 3$, how does the solution $y(t)$ behave as $t \to \infty$?

(2) Solve (possibly implicitly) each of the following initial-value problems. Identify their intervals of definition.
   (a) $\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{2ty}{1+t^2} = t^2\,, \quad y(0) = 1\,.$
   (b) $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{e^x y + 2x}{2y + e^x} = 0\,, \quad y(0) = 0\,.$

(3) Let $y(t)$ be the solution of the initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y^2 + t^2\,, \qquad y(0) = 1\,.$$

Use two steps of the forward Euler method to approximate $y(0.2)$.

(4) Give an explicit real-valued general solution of the following equations.
   (a) $\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} + 5y = te^t + \cos(2t)$
   (b) $\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 9y = \tan(3t)$

(5) When a mass of 2 kilograms is hung vertically from a spring, it stretches the spring 0.5 meters. (Gravitational acceleration is 9.8 m/sec^2.) At $t = 0$ the mass is set in motion from 0.3 meters below its equilibrium (rest) position with a upward velocity of 2 m/sec. Neglect drag and assume that the spring force is proportional to its displacement. Formulate an initial-value problem that governs the motion of the mass for $t > 0$. (DO NOT solve this initial-value problem; just write it down!)

(6) Give an explicit general solution of the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2\frac{\mathrm{d}y}{\mathrm{d}t} + 5y = 0\,.$$

Sketch a typical solution for $t \ge 0$. If this equation governs a damped spring-mass system, is the system over, under, or critically damped?

(7) Find the Laplace transform $Y(s)$ of the solution $y(t)$ to the initial-value problem

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 4\frac{\mathrm{d}y}{\mathrm{d}t} + 8y = f(t)\,, \qquad y(0) = 2\,, \quad y'(0) = 4\,.$$

where

$$f(t) = \begin{cases} 4 & \text{for } 0 \le t < 2\,, \\ t^2 & \text{for } 2 \le t\,. \end{cases}$$

You may refer to the table in Section 6.2 of the book. (DO NOT take the inverse Laplace transform to find $y(t)$; just solve for $Y(s)$!)

(8) Find the function $y(t)$ whose Laplace transform $Y(s)$ is given by

$$\text{(a)}\quad Y(s) = \frac{e^{-3s}4}{s^2 - 6s + 5}\,, \qquad \text{(b)}\quad Y(s) = \frac{e^{-2s}s}{s^2 + 4s + 8}\,.$$

You may refer to the table in Section 6.2 of the book.

(9) Consider the real vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t^3 \\ 3 + t^4 \end{pmatrix}$.

(a) Compute the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2](t)$.

(b) Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to the linear system $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x}$.

(c) Give a general solution to the system you found in part (b).

(10) Give a general real vector-valued solution of the linear planar system $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x}$ for

$$\text{(a)}\quad \mathbf{A} = \begin{pmatrix} 6 & 4 \\ 4 & 0 \end{pmatrix}\,, \qquad \text{(b)}\quad \mathbf{A} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\,.$$

(11) A real 2×2 matrix $\mathbf{A}$ has eigenvalues 2 and -1 with associated eigenvectors

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ 2 \end{pmatrix}\,.$$

(a) Give a general solution to the linear planar system $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x}$.

(b) Classify the stability of the origin. Sketch a phase-plane portrait for this system and identify its type. (Carefully mark all sketched trajectories with arrows!)

(12) Consider the nonlinear planar system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -5y\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = x - 4y - x^2\,.$$

(a) Find all of its equilibrium (critical, stationary) points.

(b) Compute the coefficient matrix of the linearization (the derivative matrix) at each equilibrium (critical, stationary) point.

(c) Classify the type and stability of each equilibrium (critical, stationary) point.

(d) Sketch a plausible global phase-plane portrait. (Carefully mark all sketched trajectories with arrows!)

(13) Consider the nonlinear planar system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(3 - 3x + 2y)\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = y(6 - x - y)\,.$$

Do parts (a-d) as for the previous problem.