Solutions to Homework Problems on the Green Function Method
Fall 2008, Math 246, Professor David Levermore

1. Compute the Green functions associated with the following differential operators L.

$$
\begin{aligned}
&\text{a)} \quad \mathrm{L} = \mathrm{D}^2 + 4\mathrm{D} - 5 \\
&\text{b)} \quad \mathrm{L} = \mathrm{D}^2 + 4\mathrm{D} + 5 \\
&\text{c)} \quad \mathrm{L} = \mathrm{D}^3 + 4\mathrm{D}
\end{aligned}
$$

Solutions: The final answers are:

$$
\begin{aligned}
&\text{a)} \quad g(t) = \frac{e^t - e^{-5t}}{6}\,, \\
&\text{b)} \quad g(t) = e^{-2t}\sin(t)\,, \\
&\text{c)} \quad g(t) = \frac{1 - \cos(2t)}{4}\,.
\end{aligned}
$$

2. Use the Green function method to find a general solution of the following equations.

$$
\begin{aligned}
&\text{a)} \quad \mathrm{D}^2 y + \mathrm{D}y - 2y = \frac{1}{e^t + 1} \\
&\text{b)} \quad \mathrm{D}^2 y + y = \frac{1}{\cos(t)} \\
&\text{c)} \quad \mathrm{D}^2 y + y = \frac{1}{9 + 16\sin(t)^2}
\end{aligned}
$$

Solutions: The final answers are:

$$
\begin{aligned}
\text{a)} \quad y &= c_1 e^t + c_2 e^{-2t} + \tfrac{1}{3} e^t \int_0^t \frac{e^{-s}}{e^s + 1}\,\mathrm{d}s - \tfrac{1}{3} e^{-2t} \int_0^t \frac{e^{2s}}{e^s + 1}\,\mathrm{d}s \\
&= c_1 e^t + c_2 e^{-2t} + \tfrac{1}{3} e^t \left[1 - e^{-t} + \log\!\left(\frac{e^{-t} + 1}{2}\right)\right] + \tfrac{1}{3} e^{-2t} \left[1 - e^t + \log\!\left(\frac{e^t + 1}{2}\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
\text{b)} \quad y &= c_1 \cos(t) + c_2 \sin(t) + \sin(t) \int_0^t \frac{\cos(s)}{\cos(s)}\,\mathrm{d}s - \cos(t) \int_0^t \frac{\sin(s)}{\cos(s)}\,\mathrm{d}s \\
&= c_1 \cos(t) + c_2 \sin(t) + \sin(t)\, t + \cos(t)\, \log(|\cos(t)|)\,,
\end{aligned}
$$

$$
\begin{aligned}
\text{c)} \quad y &= c_1 \cos(t) + c_2 \sin(t) + \sin(t) \int_0^t \frac{\cos(s)}{9 + 16\sin(s)^2}\,\mathrm{d}s - \cos(t) \int_0^t \frac{\sin(s)}{9 + 16\sin(s)^2}\,\mathrm{d}s \\
&= c_1 \cos(t) + c_2 \sin(t) + \tfrac{1}{12} \tan^{-1}\!\big(\tfrac{4}{3}\sin(t)\big) + \tfrac{1}{40} \log\!\left(\frac{5 + 4\cos(t)}{5 - 4\cos(t)}\,\frac{1}{9}\right).
\end{aligned}
$$