HIGHER-ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS I: Introduction and Homogeneous Case

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Because the presentation of this material in class will differ somewhat from that in the book, I felt that notes that closely follow the class presentation might be appreciated.

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1. Introduction

1.1: Normal Form and Solutions. An n^{th}-order linear ordinary differential equation can be brought into the linear normal form

$$\frac{d^n y}{dt^n} + a_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_{n-1}(t)\frac{dy}{dt} + a_n(t)y = f(t)\,. \tag{1.1}$$

Here $a_1(t), \cdots, a_n(t)$ are call *coefficients* while $f(t)$ is called the *forcing* or *driving*. When $f(t) = 0$ the equation is said to be *homogeneous*; otherwise it is said to be *nonhomogeneous*.

Definition: We say that $y = Y(t)$ is a *solution* of (1.1) over an interval (t_L, t_R) provided that:

- the function Y is n-times differentiable over (t_L, t_R),
- the coefficients $a_1(t)$, $a_2(t)$, $\cdots$, $a_n(t)$, and the forcing $f(t)$ are defined over (t_L, t_R),
- the equation

$$Y^{(n)}(t) + a_1(t)Y^{(n-1)}(t) + \cdots + a_{n-1}(t)Y'(t) + a_n(t)Y(t) = f(t)$$

is satisfied for every t in (t_L, t_R).

The first two bullets simply say that every term appearing in the equation is defined over the interval (t_L, t_R), while the third says the equation is satisfied at each time t in (t_L, t_R).

1.2: Initial-Value Problem. An *initial-value problem* associated with (1.1) seeks a solution $y = Y(t)$ of (1.1) that also satisfies the *initial conditions*

$$Y(t_I) = y_0\,, \qquad Y'(t_I) = y_1\,, \qquad \cdots \qquad Y^{(n-1)}(t_I) = y_{n-1}\,, \tag{1.2}$$

for some *initial time* (or *initial point*) t_I and *initial data* (or *initial values*) $y_0, y_1, \cdots, y_{n-1}$. You should know the following basic existence and uniqueness theorem about initial-value problems, which we state without proof.

Theorem 1.1 (Basic Existence and Uniqueness Theorem): Let the functions $a_1, a_2, \cdots, a_n$, and f all be continuous over an interval (t_L, t_R). Then given any initial time $t_I \in (t_L, t_R)$ and any initial data $y_0, y_1, \cdots, y_{n-1}$ there exists a unique solution $y = Y(t)$ of (1.1) that satisfies the initial conditions (1.2). Moreover, this solution has at least n continuous derivatives over (t_L, t_R). If the functions $a_1, a_2, \cdots, a_n$, and f all have k continuous derivatives over (t_L, t_R) then this solution has at least $k + n$ continuous derivatives over (t_L, t_R).

Remark: For first-order linear equations ($n = 1$) this theorem was essentially proved when we showed that the unique solution of the initial-value problem

$$\frac{dy}{dt} + a(t)y = f(t)\,, \qquad Y(t_I) = y_0\,,$$

is given by the formula

$$Y(t) = \exp\bigg(-\int_{t_I}^{t} a(s)\,ds\bigg)\bigg[y_0 + \int_{t_I}^{t} \exp\bigg(-\int_{t_I}^{s} a(s_1)\,ds_1\bigg) f(s)\,ds\bigg]. \tag{1.3}$$

Because there is no such general formula for the solution of the initial-value problem when $n \geq 2$, the proof of this theorem for higher order equations requires methods beyond the scope of this course.

Remark: Later in this chapter we will see that for special choices of coefficients one can construct explicit formulas for the solution of the initial-value problem when $n \geq 2$. Even in such cases we will appeal to this theorem to assert the uniqueness of the solution.

Remark: This theorem states the "counting fact" that solutions of any n^{th}-order linear equation are uniquely specified by n additional pieces of information — specifically, the values of the solution Y and its first $n-1$ derivatives at an initial time t_I. It is natural to ask whether one has a similar result if one replaces the n initial conditions (1.2) with any n conditions on Y. For example, can one use n conditions that specify the values of Y and some of its derivatives at more than one time? Such a problem is a so-called boundary-value problem. In general solutions to such problems either may not exist or may not be unique. In this course we shall therefore focus on initial-value problems, which are simpler. Boundary-value problems are very important and are studied in more advanced courses.

You should be able to use the Basic Existence and Uniqueness Theorem to argue that certain functions cannot be the solution of a given order of homogeneous linear ordinary differential equations. This is usually argued by contradition.

Example: $\sin(t^3)$ cannot be the solution of any equation of the form

$$\frac{d^3z}{dt^3} + a_1(t)\frac{d^2z}{dt^2} + a_2(t)\frac{dz}{dt} + a_3(t)z = 0\,,$$

where a_1, a_2, and a_3, are continuous over an open interval containing 0. Suppose otherwise — namely, suppose that $Z(t) = \sin(t^3)$ satisfies such an equation. Because

$$Z'(t) = 3t^2\cos(t^3)\,, \qquad Z''(t) = 6t\cos(t^3) - 9t^4\sin(t^3)\,.$$

we see that $Z(t)$ satisfies the equation and the initial conditions

$$Z(0) = Z'(0) = Z''(0) = 0\,.$$

However the Basic Existence and Uniqueness Theorem implies that $Z(t) = 0$ is the only solution of the equation that satisfies these initial conditions, which contradicts the fact that $Z(t) = \sin(t^3)$.

1.3: Intervals of Existence. You should also be able to use the Basic Existence and Uniqueness Theorem to identify the interval of existence for solutions of (1.1). This is done very much like the way you identified intervals of existence for solutions of first-order linear equations. Specifically, if $Y(t)$ is the solution of the initial value problem (1.1-1.2) then its interval of existence will be (t_L, t_R) whenever:

- all the coefficients and the forcing are continuous over (t_L, t_R),
- the initial time t_I is in (t_L, t_R),
- either a coefficient or the forcing is not defined at each of $t = t_L$ and $t = t_R$.

This is because the first two bullets along with the Basic Existence and Uniqueness Theorem imply that the interval of existence will be at least (t_L, t_R), while the last two bullets along with our definition of solution imply that the interval of existence can be no bigger than (t_L, t_R) because the equation breaks down at $t = t_L$ and $t = t_R$. This argument works when $t_L = -\infty$ or $t_R = \infty$.

Remark: This does not mean that every solution of (1.1) will become singular at either $t = t_L$ or $t = t_R$ when those endpoints are finite..

Example: Consider the initial value problem

$$\frac{d^3x}{dt^3} + \frac{1}{t^2 - 4}x = \cos(t)\,, \qquad x(1) = 3\,, \quad x'(1) = 0\,, \quad x''(1) = 0\,.$$

The coefficient and forcing are continuous over $(-2, 2)$; the initial time is $t = 1$, which is in $(-2, 2)$; and the coefficient is not defined at $t = -2$ and at $t = 2$. The interval of existence of the solution is therefore $(-2, 2)$.

Example: Consider the initial value problem

$$\frac{d^4y}{dt^4} + \frac{1}{t-4}\frac{dy}{dt} = \frac{e^t}{2+t}\,, \qquad y(0) = y'(0) = y''(0) = y'''(0) = 0\,.$$

The coefficient and forcing are continuous over $(-2, 4)$; the initial time is $t = 0$, which is in $(-2, 4)$; the coefficient is not defined at $t = 4$ while the forcing is not defined at $t = -2$. The interval of existence of the solution is therefore $(-2, 4)$.

Example: Consider the initial value problem

$$\frac{d^4y}{dt^4} + \frac{1}{t-4}\frac{dy}{dt} = \frac{e^t}{2+t}\,, \qquad y(6) = y'(6) = y''(6) = y'''(6) = 0\,.$$

The coefficient and forcing are continuous over $(4, \infty)$; the initial time is $t = 6$, which is in $(4, \infty)$; the coefficient is not defined at $t = 4$. The interval of existence of the solution is therefore $(4, \infty)$.

2. Homogeneous Equations: General Theory

2.1: Linear Superposition. Before we examine the general case, we study the special case of homogeneous linear equations. These have the normal form

$$\frac{d^n y}{dt^n} + a_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_{n-1}(t)\frac{dy}{dt} + a_n(t)y = 0\,. \tag{2.1}$$

We will assume throughout this section that the coefficients $a_1, a_2, \cdots, a_n$ are continuous over an interval (t_L, t_R), so that Therorem 1.1 can be applied. We will exploit the following property of homogeneous equations.

> **Theorem 2.1 (Linear Superposition):** If $Y_1(t)$ and $Y_2(t)$ are solutions of (2.1) then so is
>
> $$c_1 Y_1(t) + c_2 Y_2(t)\,,$$
>
> for any values of the constants c_1 and c_2. More generally, if $Y_1(t)$, $Y_2(t)$, $\cdots$, $Y_m(t)$ are solutions of (2.1) then so is
>
> $$c_1 Y_1(t) + c_2 Y_2(t) + \cdots + c_m Y_m(t)\,,$$
>
> for any values of the constants c_1, c_2, $\cdots$, c_m.

Remark: This theorem states that any linear combination of solutions of (2.1) is also a solution of (2.1). It thereby provides a way to construct a whole family of solutions from a finite number of them.

Suppose you know n "different" solutions of (2.1), $Y_1(t)$, $Y_2(t)$, $\cdots$, $Y_n(t)$. It is natural to ask if you can construct the solution of the initial-value problem as a linear combination of $Y_1(t)$, $Y_2(t)$, $\cdots$, $Y_n(t)$. Set

$$Y(t) = c_1 Y_1(t) + c_2 Y_2(t) + \cdots + c_n Y_n(t)\,.$$

By the superposition theorem this is a solution of (2.1). One only has to check that values of c_1, c_2, $\cdot$, c_n can be found such that $Y(t)$ satisfies the initial conditions

$$\begin{aligned}
y_0 = {} & Y(t_I) = c_1 Y_1(t_I) + c_2 Y_2(t_I) + \cdots + c_n Y_n(t_I)\,,\\
y_1 = {} & Y'(t_I) = c_1 Y_1'(t_I) + c_2 Y_2'(t_I) + \cdots + c_n Y_n'(t_I)\,,\\
\vdots \quad & \quad \vdots\\
y_{n-1} = {} & Y^{(n-1)}(t_I) = c_1 Y_1^{(n-1)}(t_I) + c_2 Y_2^{(n-1)}(t_I) + \cdots + c_n Y_n^{(n-1)}(t_I)\,.
\end{aligned} \tag{2.2}$$

This is a system of n linear algebraic equations for the n unknowns c_1, c_2, $\cdot$, c_n. It seems likely that one can often solve this system.

Example: One can check that e^{2t} and e^{-t} are solutions of

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0\,.$$

Let's find c_1 and c_2 such that $Y(t) = c_1 e^{2t} + c_2 e^{-t}$ satifies the initial conditions

$$Y(0) = y_0\,, \qquad Y'(0) = y_1\,.$$

Because $Y'(t) = c_1 2e^{2t} - c_2 e^{-t}$, these initial condition become

$$y_0 = c_1 + c_2\,, \qquad y_1 = 2c_1 - c_2\,.$$

These can be solved to find

$$c_1 = \frac{y_0 + y_1}{3}\,, \qquad c_2 = \frac{2y_0 - y_1}{3}\,.$$

Hence, for any choice of y_0 and y_1 the solution of the initial value problem is given by

$$Y(t) = \frac{y_0 + y_1}{3}\, e^{2t} + \frac{2y_0 - y_1}{3}\, e^{-t}\,.$$

Example: One can check that $\cos(2t)$ and $\sin(2t)$ are solutions of

$$\frac{d^2y}{dt^2} + 4y = 0\,.$$

Let's find c_1 and c_2 such that $Y(t) = c_1 \cos(2t) + c_2 \sin(2t)$ satifies the initial conditions

$$Y(0) = y_0\,, \qquad Y'(0) = y_1\,.$$

Because $Y'(t) = -2c_1 \sin(2t) + 2c_2 \cos(2t)$, these initial condition become

$$y_0 = c_1\,, \qquad y_1 = 2c_2\,.$$

These can be easily solved to find

$$c_1 = y_0\,, \qquad c_2 = \frac{y_1}{2}\,.$$

Hence, for any choice of y_0 and y_1 the solution of the initial value problem is given by

$$Y(t) = y_0 \cos(2t) + y_1 \frac{\sin(2t)}{2}\,.$$

Example: One can check that t and $t^2 - 1$ are solutions of

$$(1 + t^2)\frac{d^2y}{dt^2} - 2t\,\frac{dy}{dt} + 2y = 0\,.$$

Let's find c_1 and c_2 such that $Y(t) = c_1 t + c_2(t^2 - 1)$ satifies the initial conditions

$$Y(1) = y_0\,, \qquad Y'(1) = y_1\,.$$

Because $Y'(t) = c_1 + 2c_2 t$, these initial condition become

$$y_0 = c_1\,, \qquad y_1 = c_1 + 2c_2\,.$$

These can be solved to find

$$c_1 = y_0\,, \qquad c_2 = \frac{y_1 - y_0}{2}\,.$$

Hence, for any choice of y_0 and y_1 the solution of the initial value problem is given by

$$Y(t) = y_0 t + \frac{y_1 - y_0}{2}(t^2 - 1)\,.$$

Example: One can check that e^{4t}, e^{3t}, and e^{-t} are solutions of

$$\frac{d^3y}{dt^3} - 6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 12y = 0\,.$$

Let's find c_1, c_2, and c_3 such that $Y(t) = c_1 e^{4t} + c_2 e^{3t} + c_3 e^{-t}$ satifies the initial conditions

$$Y(0) = y_0\,, \qquad Y'(0) = y_1\,, \qquad Y''(0) = y_2\,.$$

Because

$$\begin{aligned} Y'(t) &= c_1 4e^{4t} + c_2 3e^{3t} - c_3 e^{-t}\,, \\ Y''(t) &= c_1 16e^{4t} + c_2 9e^{3t} + c_3 e^{-t}\,, \end{aligned}$$

these initial condition become

$$\begin{aligned} y_0 &= c_1 + c_2 + c_3\,, \\ y_1 &= 4c_1 + 3c_2 - c_3\,, \\ y_2 &= 16c_1 + 9c_2 + c_3\,. \end{aligned}$$

These can be solved to find

$$c_1 = \frac{-3y_0 - 2y_1 + y_2}{5}\,, \qquad c_2 = \frac{4y_0 + 3y_1 - y_2}{4}\,, \qquad c_3 = \frac{12y_0 - 7y_1 + y_2}{20}\,.$$

Hence, for any choice of y_0, y_1, and y_2 the solution of the initial value problem is given by

$$Y(t) = \frac{-3y_0 - 2y_1 + y_2}{5}\,e^{4t} + \frac{4y_0 + 3y_1 - y_2}{4}\,e^{3t} + \frac{12y_0 - 7y_1 + y_2}{20}\,e^{-t}\,.$$

2.2: Wronskians. System (2.2) will have a unique solution for every set of initial data $y_0, y_1, \cdots, y_{n-1}$ if and only if

$$\det \begin{pmatrix} Y_1(t_I) & Y_2(t_I) & \cdots & Y_n(t_I) \\ Y_1'(t_I) & Y_2'(t_I) & \cdots & Y_n'(t_I) \\ \vdots & \vdots & \ddots & \vdots \\ Y_1^{(n-1)}(t_I) & Y_2^{(n-1)}(t_I) & \cdots & Y_n^{(n-1)}(t_I) \end{pmatrix} \neq 0 \,. \tag{2.3}$$

This follows from Theorem A.1 which is given in Appendix A on linear algebraic systems. In this section we explore this condition further. We begin with a definition.

Definition: Given any n functions $Y_1, Y_2, \cdots, Y_n$ that are $n-1$ times differentiable over an interval (t_L, t_R), a new function $W[Y_1, Y_2, \cdots, Y_n]$, called the *Wronskian* of $Y_1, Y_2, \cdots, Y_n$, is defined over (t_L, t_R) by

$$W[Y_1, Y_2, \cdots, Y_n](t) = \det \begin{pmatrix} Y_1(t) & Y_2(t) & \cdots & Y_n(t) \\ Y_1'(t) & Y_2'(t) & \cdots & Y_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ Y_1^{(n-1)}(t) & Y_2^{(n-1)}(t) & \cdots & Y_n^{(n-1)}(t) \end{pmatrix} \,. \tag{2.4}$$

Condition (2.3) can then be recast as simply

$$W[Y_1, Y_2, \cdots, Y_n](t_I) \neq 0 \,. \tag{2.5}$$

It is natural to ask whether condition (2.5) can hold for some initial time t_I but not for other times. The following result will allow us to show that this is not the case.

Theorem 2.2 (Abel's Theorem): If $Y_1, Y_2, \cdots, Y_n$ are solutions of (2.1) then $W[Y_1, Y_2, \cdots, Y_n]$ satisfies the first-order linear equation

$$\frac{d}{dt} W[Y_1, Y_2, \cdots, Y_n] + a_1(t) W[Y_1, Y_2, \cdots, Y_n] = 0 \,, \tag{2.6}$$

whereby formula (1.3) implies that

$$W[Y_1, Y_2, \cdots, Y_n](t) = W[Y_1, Y_2, \cdots, Y_n](t_I) \, \exp\left(- \int_{t_I}^{t} a_1(s) \, ds \right) . \tag{2.7}$$

Proof for Second Order Case: We will not give a proof of Abel's Theorem in its general setting because to do so would require more properties of determinants than we will cover in this course. We will however give a proof for the second order case, which is the case that you will encounter most often in this course.

Let Y_1 and Y_2 be two solutions of the second order homogeneous linear equation

$$\frac{d^2y}{dt^2} + a_1(t)\frac{dy}{dt} + a_2(t)y = 0\,.$$

Their Wronskian is given by

$$W[Y_1, Y_2](t) = \det\begin{pmatrix} Y_1(t) & Y_2(t) \\ Y_1'(t) & Y_2'(t) \end{pmatrix} = Y_1(t)Y_2'(t) - Y_1'(t)Y_2(t)\,.$$

Differentiating this formula and then using the differential equation to eliminate $Y_1''(t)$ and $Y_2''(t)$ yields

$$\begin{aligned}
\frac{d}{dt}W[Y_1, Y_2](t) &= Y_1'(t)Y_2'(t) + Y_1(t)Y_2''(t) - Y_1'(t)Y_2'(t) - Y_1''(t)Y_2(t) \\
&= Y_1(t)Y_2''(t) - Y_1''(t)Y_2(t) \\
&= Y_1(t)\big(- a_1(t)Y_2'(t) - a_2(t)Y_2(t)\big) - \big(- a_1(t)Y_1'(t) - a_2(t)Y_1(t)\big)Y_2(t) \\
&= -a_1(t)\big(Y_1(t)Y_2'(t) - Y_1'(t)Y_2(t)\big) - a_2(t)\big(Y_1(t)Y_2(t) - Y_1(t)Y_2(t)\big) \\
&= -a_1(t)W[Y_1, Y_2](t)\,,
\end{aligned}$$

which is equivalent to the first-order equation (2.6) asserted in Abel's Theorem. □

Exercise: Give a proof of Abel's Theorem for the third order case along the lines of the one above for the second order case.

An important consequence of Abel's Theorem is that the Wronskian of n solutions of (2.1) is either always zero or never zero.

> **Theorem 2.3:** If $Y_1, Y_2, \cdots, Y_n$ are solutions of (2.1) over an interval (t_L, t_R) then their Wronskian $W[Y_1, Y_2, \cdots, Y_n]$ is either zero everywhere in (t_L, t_R) or zero nowhere in (t_L, t_R).

Proof: Suppose that $W[Y_1, Y_2, \cdots, Y_n](t_I) = 0$ for some t_I in (t_L, t_R). Then formula (2.7) immediately implies that $W[Y_1, Y_2, \cdots, Y_n](t) = 0$ everywhere in (t_L, t_R). On the other hand, suppose that $W[Y_1, Y_2, \cdots, Y_n](t_I) \neq 0$ for some t_I in (t_L, t_R). Then because the exponential factor in formula (2.7) is always positive, the formula implies that $W[Y_1, Y_2, \cdots, Y_n](t) \neq 0$ everywhere in (t_L, t_R). □

2.3: Fundamental Sets of Solutions and General Solutions. Theorem 2.3 shows us that either condition (2.5) holds everywhere $Y_1, Y_2, \cdots, Y_n$ are defined, or it holds nowhere. This means that when the Wronskian $W[Y_1, Y_2, \cdots, Y_n]$ is nonzero you can always find the unique solution of any initial value problem for any initial time t_I and any initial data y_0, y_1, $\cdots$, y_n. This fact motivates the following definition.

> **Definition:** A set of n solutions of an n^{th} order homogeneous linear ordinary differential equation is said to be *fundamental* if its Wronskian is nonzero.

The importance of this concept is evident in the following.

Theorem 2.4: Let Y_1, Y_2, $\cdots$, Y_n be a fundamental set of solutions of equation (2.1) over the interval (t_L, t_R). Then every solution of (2.1) over the interval (t_L, t_R) can be expressed as a unique linear combination of Y_1, Y_2, $\cdots$, Y_n.

Proof: Let $Y(t)$ be any solution of (2.1) over (t_L, t_R). Consider the n-parameter family

$$c_1 Y_1(t) + c_2 Y_2(t) + \cdots + c_n Y_n(t)\,. \tag{2.8}$$

Because $Y_1, Y_2, \cdots, Y_n$ is a fundamental set of solutions, we know $W[Y_1, Y_2, \cdots, Y_n](t_I) \neq 0$ for any time t_I in (t_L, t_R). There is therefore a unique set of values for c_1, c_2, $\cdots$, c_n such that (2.8) will match the initial values $Y(t_I)$, $Y'(t_I)$, $\cdots$, $Y^{(n-1)}(t_I)$. Because both $Y(t)$ and (2.8) for these values of c_1, c_2, $\cdots$, c_n are solutions of (2.1) and they satisfy the same initial values at t_I, the uniqueness assertion of Theorem 1.1 implies they are equal. □

Theorem 2.4 motivates the following definition.

Definition: If Y_1, Y_2, $\cdots$, Y_n is a fundamental set of solutions of an n^{th} order homogeneous linear ordinary differential equation then the n-parameter family (2.8) is called a *general solution* of the equation.

Example: You can check that $Y_1(t) = e^{2t}$ and $Y_2(t) = e^{-t}$ are solutions of

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0\,.$$

They are a fundamental set of solutions because

$$\begin{aligned} W[Y_1, Y_2](t) &= \det\begin{pmatrix} Y_1(t) & Y_2(t) \\ Y_1'(t) & Y_2'(t) \end{pmatrix} = \det\begin{pmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{pmatrix} \\ &= -e^{2t}e^{-t} - 2e^{2t}e^{-t} = -3e^{t} \neq 0\,. \end{aligned}$$

A general solution is therefore $c_1 e^{2t} + c_2 e^{-t}$.

Example: You can check that $Y_1(t) = \cos(2t)$ and $Y_2(t) = \sin(2t)$ are solutions of

$$\frac{d^2y}{dt^2} + 4y = 0\,.$$

They are a fundamental set of solutions because

$$\begin{aligned} W[Y_1, Y_2](t) &= \det\begin{pmatrix} Y_1(t) & Y_2(t) \\ Y_1'(t) & Y_2'(t) \end{pmatrix} = \det\begin{pmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{pmatrix} \\ &= 2\cos(2t)^2 + 2\sin(2t)^2 = 2 \neq 0\,. \end{aligned}$$

A general solution is therefore $c_1 \cos(2t) + c_2 \sin(2t)$.

Example: You can check that $Y_1(t) = t$ and $Y_2(t) = t^2 - 1$ are solutions of

$$(1 + t^2)\frac{d^2y}{dt^2} - 2t\,\frac{dy}{dt} + 2y = 0\,.$$

They are a fundamental set of solutions because

$$W[Y_1, Y_2](t) = \det\begin{pmatrix} Y_1(t) & Y_2(t) \\ Y_1'(t) & Y_2'(t) \end{pmatrix} = \det\begin{pmatrix} t & t^2 - 1 \\ 1 & 2t \end{pmatrix} = 2t^2 - (t^2 - 1) = t^2 + 1 \neq 0\,.$$

A general solution is therefore $c_1 t + c_2(t^2 - 1)$.

Example: You can check that $Y_1(t) = e^{4t}$, $Y_2(t) = e^{3t}$, and $Y_3(t) = e^{-t}$ are solutions of

$$\frac{d^3y}{dt^3} - 6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 12y = 0\,.$$

They are a fundamental set of solutions because

$$\begin{aligned} W[Y_1, Y_2, Y_3](t) &= \det\begin{pmatrix} Y_1(t) & Y_2(t) & Y_3(t) \\ Y_1'(t) & Y_2'(t) & Y_3'(t) \\ Y_1''(t) & Y_2''(t) & Y_3''(t) \end{pmatrix} = \det\begin{pmatrix} e^{4t} & e^{3t} & e^{-t} \\ 4e^{4t} & 3e^{3t} & -e^{-t} \\ 16e^{4t} & 9e^{3t} & e^{-t} \end{pmatrix} \\ &= 3e^{4t}e^{3t}e^{-t} - 16e^{4t}e^{3t}e^{-t} + 36e^{4t}e^{3t}e^{-t} \\ &\quad - 48e^{4t}e^{3t}e^{-t} + 9e^{4t}e^{3t}e^{-t} - 4e^{4t}e^{3t}e^{-t} \\ &= (3 - 16 + 36 - 48 + 9 - 4)e^{6t} = -20e^{6t} \neq 0\,. \end{aligned}$$

A general solution is therefore $c_1 e^{4t} + c_2 e^{3t} + c_3 e^{-t}$.

2.4: Linear Independence of Solutions. In the last setion we defined fundamental sets of solutions of a homogeneous linear ordinary differential equation in terms of nonzero Wronskians. Here we develop another characterization of fundamental sets of solutions based on the following notions.

> **Definition:** Functions Y_1, Y_2, $\cdots$, Y_m defined over an interval (t_L, t_R) are said to be *linearly dependent* if there exists constants c_1, c_2, $\cdots$, c_m, not all zero, such that
>
> $$0 = c_1 Y_1(t) + c_2 Y_2(t) + \cdots + c_m Y_m(t) \quad \text{for every } t \text{ in } (t_L, t_R)\,. \tag{2.9}$$
>
> Otherwise they are said to be *linearly independent.*

If Y_1, Y_2, $\cdots$, Y_m are linearly dependent then for any c_k that is nonzero, one can solve (2.9) for $Y_k(t)$ as a linear combination of the other functions. For example, if $c_1 \neq 0$ then

$$Y_1(t) = -\frac{c_2}{c_1}\,Y_2(t) - \cdots - \frac{c_m}{c_1}\,Y_m(t) \quad \text{for every } t \text{ in } (t_L, t_R)\,.$$

Because there is at least one nonzero c_k, this can always be done for some Y_k.

Example: The functions $\cos(2t)$, $\cos(t)^2$ and 1 are linearly dependent over $(-\infty, \infty)$ because

$$\cos(2t) = \cos(t)^2 - \sin(t)^2 = 2\cos(t)^2 - 1\,.$$

Remark: If one of the functions Y_1, Y_2, $\cdots$, Y_m is identically zero over (t_L, t_R) then the set is linearly dependent. For example, suppose that $Y_1(t) = 0$ for every t in (t_L, t_R). Then (2.9) holds with $c_1 = 1$ and $c_2 = \cdots = c_m = 0$.

Remark: Two functions Y_1 and Y_2, neither of which is identically zero, are linearly dependent if and only if they are proportional to each other.

Example: The functions t and t^2 are linearly independent over $(0, 1)$ because they are not proportional to each other. If you think graphically then there is clearly no constant k such that $t^2 = kt$ for every t in $(0, 1)$ because the parabola $y = t^2$ is not a line. Hence, these functions are not linearly dependent.

A good way to generally approach establishing linear independence is the following. A set functions Y_1, Y_2, $\cdots$, Y_m defined over an interval (t_L, t_R) is linearly independent if the linear relation (2.9) can only hold when $c_1 = c_2 = \cdots = c_m = 0$. When a set of functions is linearly independent there are many ways to show this.

Example: The functions 1, t and t^2 are linearly independent over $(-\infty, \infty)$. We show this by supposing the linear relation

$$0 = c_1 + c_2 t + c_3 t^2 \qquad \text{for every } t \text{ in } (-\infty, \infty)\,.$$

If we set $t = 0$, $t = 1$, and $t = -1$ into this relation, we obtain the linear algebraic system

$$\begin{aligned} 0 &= c_1\,, \\ 0 &= c_1 + c_2 + c_3\,, \\ 0 &= c_1 - c_2 + c_3\,. \end{aligned}$$

This can be easily solved to show that $c_1 = c_2 = c_3 = 0$, whereby you conclude that 1, t and t^2 are linearly independent. A similar argument works if you had chosen to evaluate the linear relations at any other three distinct points, say $t = 2$, $t = 4$, and $t = 6$. We chose to use $t = 0$, $t = 1$, and $t = -1$ because they led to a simple linear algebraic system.

An alternative approach to the above example is to differentiate the linear relation twice with respect to t, thereby obtaining

$$\begin{aligned} 0 &= c_1 + c_2 t + c_3 t^2\,, \\ 0 &= c_2 + 2c_3 t\,, \qquad\qquad \text{for every } t \text{ in } (-\infty, \infty)\,. \\ 0 &= 2c_3\,, \end{aligned}$$

If we set $t = 0$ into these equations we immediately see that $c_1 = c_2 = c_3 = 0$, whereby we conclude that 1, t and t^2 are linearly independent. We can generalize this approach as follows.

Theorem 2.5: If Y_1, Y_2, $\cdots$, Y_m is a set of $m-1$ times differentiable functions over an interval (t_L, t_R) such that $W[Y_1, Y_2, \cdots, Y_m](t_I) \neq 0$ for some t_I in (t_L, t_R) then they are linearly independent.

Proof: We show this by supposing the linear relation

$$0 = c_1 Y_1(t) + c_2 Y_2(t) + \cdots + c_m Y_m(t) \quad \text{for every } t \text{ in } (t_L, t_R)\,.$$

If we differentiate this relation $m-1$ times with respect to t and evaluate the resulting relationships at $t = t_I$, we obtain the linear algebraic system

$$\begin{aligned} 0 = \quad & c_1 Y_1(t_I) \quad + \quad c_2 Y_2(t_I) \quad + \cdots + \quad c_m Y_m(t_I)\,, \\ 0 = \quad & c_1 Y_1'(t_I) \quad + \quad c_2 Y_2'(t_I) \quad + \cdots + \quad c_m Y_m'(t_I)\,, \\ & \vdots \\ 0 = {} & c_1 Y_1^{(m-1)}(t_I) + c_2 Y_2^{(m-1)}(t_I) + \cdots + c_m Y_m^{(m-1)}(t_I)\,. \end{aligned}$$

Because $W[Y_1, Y_2, \cdots, Y_m](t_I) \neq 0$, it follows from Theorem A.1 of Appendix A that $c_1 = c_2 = \cdots = c_m = 0$ is the only solution to this system, from which we conclude the functions Y_1, Y_2, $\cdots$, Y_m are linearly independent. $\square$

It is natural to ask if linear independence implies having a Wronskian that is nonzero somewhere (or what is the same, if having a Wronskian that is zero everywhere implies linear dependence.) The following example shows that this is not the case.

Example: Let $Y_1(t) = t^2$ and $Y_2(t) = |t|t$ over $(-\infty, \infty)$. Because $Y_1'(t) = 2t$ and $Y_2'(t) = 2|t|$ over $(-\infty, \infty)$, we have

$$\begin{aligned} W[Y_1, Y_2](t) &= \det\begin{pmatrix} Y_1(t) & Y_2(t) \\ Y_1'(t) & Y_2'(t) \end{pmatrix} = \det\begin{pmatrix} t^2 & |t|t \\ 2t & 2|t| \end{pmatrix} \\ &= 2|t|t^2 - 2|t|t^2 = 0 \qquad \text{for every } t \text{ in } (-\infty, \infty)\,. \end{aligned}$$

However, it is clear that Y_1 and Y_2 are not porportional, and therefore are linearly independent even though their Wronskian is zero everywhere. Alternatively, you could argue they are linearly independent by first supposing the linear relation

$$0 = c_1 t^2 + c_2 |t|t \qquad \text{for every } t \text{ in } (-\infty, \infty)\,.$$

If we set $t = 1$ and $t = -1$ into this relation, we obtain the linear algebraic system

$$0 = c_1 + c_2\,, \qquad 0 = c_1 - c_2\,.$$

This can be easily solved to show that $c_1 = c_2 = 0$, whereby you conclude that Y_1 and Y_2 are linearly independent.

The above example shows that a set of linearly independent functions can have a Wronskian that is zero everywhere. However, as the following theorem shows, this cannot happen for sets of n solutions of an n^{th} order homogeneous linear ordinary differential equation.

Theorem 2.6: If $Y_1, Y_2, \cdots, Y_n$ are solutions of (2.1) over an interval (t_L, t_R) then the following properties are equivalent:

(i) $W[Y_1, Y_2, \cdots, Y_n]$ is nonzero everywhere in (t_L, t_R),

(ii) $W[Y_1, Y_2, \cdots, Y_n]$ is nonzero somewhere in (t_L, t_R),

(iii) $Y_1, Y_2, \cdots, Y_n$ are linearly independent.

Remark: This is the same as saying that following properties are equivalent:

(i′) $W[Y_1, Y_2, \cdots, Y_n]$ is zero somewhere in (t_L, t_R),

(ii′) $W[Y_1, Y_2, \cdots, Y_n]$ is zero everywhere in (t_L, t_R),

(iii′) $Y_1, Y_2, \cdots, Y_n$ are linearly dependent.

The above properties are simply the negations of (i), (ii), and (iii) respectively.

Remark: This theorem shows that properties (i), (ii), and (iii) are all equivalent to $Y_1, Y_2, \cdots, Y_n$ being a fundamental set of solutions to (2.1). The equivalence of (i) and (ii) was already established by Theorem 2.3. Below we give an alternative proof of this fact.

Proof: It is clear that (i) implies (ii). The fact that (ii) implies (iii) is just Theorem 2.5. Neither of these implications requires the hypothesis that $Y_1, Y_2, \cdots, Y_n$ are solutions of (2.1). All that remains to be proved is that (iii) implies (i). We will do this by contradiction.

Suppose that $Y_1, Y_2, \cdots, Y_n$ are linearly independent and $W[Y_1, Y_2, \cdots, Y_n](t_I) = 0$ for some t_I in (t_L, t_R). Because

$$\det\begin{pmatrix} Y_1(t_I) & Y_2(t_I) & \cdots & Y_n(t_I) \\ Y_1'(t_I) & Y_2'(t_I) & \cdots & Y_n'(t_I) \\ \vdots & \vdots & \ddots & \vdots \\ Y_1^{(n-1)}(t_I) & Y_2^{(n-1)}(t_I) & \cdots & Y_n^{(n-1)}(t_I) \end{pmatrix} = W[Y_1, Y_2, \cdots, Y_n](t_I) = 0\,,$$

Theorem A.2 of Appendix A implies that the linear algebraic system

$$\begin{aligned} 0 &= c_1 Y_1(t_I) + c_2 Y_2(t_I) + \cdots + c_n Y_n(t_I)\,, \\ 0 &= c_1 Y_1'(t_I) + c_2 Y_2'(t_I) + \cdots + c_n Y_n'(t_I)\,, \\ &\vdots \\ 0 &= c_1 Y_1^{(n-1)}(t_I) + c_2 Y_2^{(n-1)}(t_I) + \cdots + c_n Y_n^{(n-1)}(t_I)\,, \end{aligned} \tag{2.10}$$

has a nonzero solution c_1, c_2, $\cdots$, c_n. Now define

$$Y(t) = c_1 Y_1(t) + c_2 Y_2(t) + \cdots + c_n Y_n(t)\,. \tag{2.11}$$

Because $Y_1, Y_2, \cdots, Y_n$ are solutions of (2.1), Theorem 2.1 (Superposition) implies that Y is also a solution of (2.1). By (2.10) we see that Y satisfies the initial conditions

$$Y(t_I) = 0\,, \qquad Y'(t_I) = 0\,, \qquad \cdots \qquad Y^{(n-1)}(t_I) = 0\,.$$

The uniqueness assertion of Theorem 1.1 then implies that $Y(t) = 0$ for every t in (t_L, t_R). Hence, by (2.11) we have

$$0 = c_1 Y_1(t) + c_2 Y_2(t) + \cdots + c_n Y_n(t)\,,$$

where the c_1, c_2, $\cdots$, c_n are not all zero. But this implies that $Y_1, Y_2, \cdots, Y_n$ are linearly dependent, which is a contradiction. We therefore conclude that $W[Y_1, Y_2, \cdots, Y_n]$ is nonzero everywhere in (t_L, t_R), thereby showing that (iii) implies (i). $\square$

3. Homogeneous Equations with Constant Coefficients

3.1: Characteristic Polynomials and the Key Identity. In Section 2 we saw how to construct general solutions of homogeneous linear differential equations given a fundamental set of solutions. While there is no general recipe for constructing fundamental sets of solutions, it can be done for special cases. Here we will study the most important such special case — namely, the case where the coefficients are constants. In that case (2.1) becomes

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy}{dt} + a_n y = 0\,, \tag{3.1}$$

where $a_1, a_2, \cdots, a_n$ are constants. We can express this compactly as

$$\mathrm{L}y = 0\,, \tag{3.2}$$

where L is the n^{th} order differential operator

$$\mathrm{L} = \frac{d^n}{dt^n} + a_1 \frac{d^{n-1}}{dt^{n-1}} + \cdots + a_{n-1} \frac{d}{dt} + a_n\,. \tag{3.3}$$

We will sometimes write

$$\mathrm{L} = p(\mathrm{D})\,, \quad \text{where} \quad \mathrm{D} = \frac{d}{dt}\,,$$

and $p(z)$ is the n^{th} degree real polynomial

$$p(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n\,. \tag{3.4}$$

This is the *characteristic polynomial* associated with the n^{th} order differential operator L.

Repeated differentiation of the function e^{zt} yields the identities

$$\frac{d}{dt} e^{zt} = z\, e^{zt}\,, \qquad \frac{d^2}{dt^2} e^{zt} = z^2\, e^{zt}\,, \qquad \cdots \qquad \frac{d^k}{dt^k} e^{zt} = z^k\, e^{zt}\,, \qquad \cdots\,,$$

for every positive integer k. Hence, we find that

$$\begin{aligned} p(\mathrm{D}) e^{zt} &= \frac{d^n}{dt^n} e^{zt} + a_1 \frac{d^{n-1}}{dt^{n-1}} e^{zt} + \cdots + a_{n-1} \frac{d}{dt} e^{zt} + a_n e^{zt} \\ &= z^n e^{zt} + a_1 z^{n-1} e^{zt} + \cdots + a_{n-1} z\, e^{zt} + a_n e^{zt} = p(z)\, e^{zt}\,, \end{aligned}$$

We have derived the KEY identity

$$\mathrm{L} e^{zt} = p(\mathrm{D}) e^{zt} = p(z)\, e^{zt}\,. \tag{3.5}$$

In the remainder of this section we will show how to use the KEY identity to find a recipe for a general solution of homogeneous equation $\mathrm{L}y = 0$.

3.2: Real Roots of Characteristic Polynomials. If r is a real root of $p(z)$ (i.e. if $p(r) = 0$) then the KEY identity (3.5) implies that

$$\mathrm{L}e^{rt} = p(r)e^{rt} = 0\,,$$

whereby e^{rt} is a solution of the homogeneous equation $\mathrm{L}y = 0$.

It follows from the above observation that if a characteristic polynomial has n simple real roots $r_1,\ r_2,\ \cdots,\ r_n$ then one has n solutions of the homogeneous equation $\mathrm{L}y = 0$. It can be shown that these solutions are independent. For example, when $n = 3$ one sees that the Wronskian is

$$\det\begin{pmatrix} e^{r_1 t} & e^{r_2 t} & e^{r_3 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} & r_3 e^{r_3 t} \\ r_1^2 e^{r_1 t} & r_2^2 e^{r_2 t} & r_3^2 e^{r_3 t} \end{pmatrix} = (r_3 - r_2)(r_2 - r_1)(r_3 - r_1)e^{(r_1+r_2+r_3)t} \neq 0\,.$$

The argument for independence when $n \geq 4$ goes similarly, but will not be given here because it is more complicated. Given this independence however, one thereby concludes that a general solution of $\mathrm{L}y = 0$ is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{r_n t}\,.$$

The most difficult part of applying this recipe is often finding the roots of the characteristic polynomial. Of course, for quadratic polynomials this can be done by completing the square or by using the quadratic formula. In this course characteristic polynomials of degree three or more will generally have some easily found root like 0, ± 1, ± 2, or ± 3. If the coefficients of $p(z)$ are integers, you should first check for roots that are factors of a_n. Given that you have found a real root r, the characteristic polynomial can be factored as

$$p(z) = (z - r)q(z)\,,$$

where $q(z)$ is an $(n-1)^{th}$ degree real polynomial

$$q(z) = z^{n-1} + b_1 z^{n-2} + \cdots + b_{n-2}z + b_{n-1}\,.$$

One thereby reduces the problem of finding roots of $p(z)$ to finding roots of $q(z)$. If a characteristic polynomial has n simple real roots $r_1,\ r_2,\ \cdots,\ r_n$ then this procedure is repeated until you have completely factored $p(z)$ into the form

$$p(z) = (z - r_1)(z - r_2)\cdots(z - r_n)\,.$$

Of course, if $p(z)$ is given to you in factored form, you can just read off the roots!

Example: Find a general solution of

$$\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0\,.$$

The characteristic polynomial is

$$p(z) = z^3 + 2z^2 - z - 2 = (z-1)(z+1)(z+2)\,.$$

Its three roots are 1, -1, -2. The solution associated with the root 1 is e^t. The solution associated with the root -1 is e^{-t}. The solution associated with the root -2 is e^{-2t}. A general solution is therefore

$$y = c_1 e^t + c_2 e^{-t} + c_3 e^{-2t}\,.$$

Of course, polynomials of degree n do not generally have n simple real roots. There are two ways this can fail to happen. First, a real root might not be simple — that is, a real root might have multiplicity greater than one. Second, polynomials might have irreducible factors which correspond to complex roots. We first examine how to treat cases with real roots of multiplicity greater than one.

Recall that r is a double real root of $p(z)$ when $(z-r)^2$ is a factor of $p(z)$. This is equivalent to the condition $p(r) = p'(r) = 0$. Differentiation of the KEY identity (3.5) with respect to z gives

$$\mathrm{L}\big(t\,e^{zt}\big) = p(z)\,t\,e^{zt} + p'(z)\,e^{zt}\,.$$

Evaluating this at $z = r$ shows that

$$\mathrm{L}\big(t\,e^{rt}\big) = p(r)\,t\,e^{rt} + p'(r)\,e^{rt} = 0\,.$$

Hence, e^{rt} and $t\,e^{rt}$ are solutions of the homogeneous equation $\mathrm{L}y = 0$. Because

$$\frac{d}{dt}e^{rt} = r\,e^{rt}\,, \qquad \frac{d}{dt}(t\,e^{rt}) = rt\,e^{rt} + e^{rt}\,,$$

the Wronskian of these solutions is

$$\det\begin{pmatrix} e^{rt} & t\,e^{rt} \\ r\,e^{rt} & rt\,e^{rt} + e^{rt} \end{pmatrix} = e^{rt}(rt\,e^{rt} + e^{rt}) - t\,e^{rt} r\,e^{rt} = e^{2rt} \neq 0\,.$$

These solutions are therefore independent.

Recall that r is a triple real root of $p(z)$ when $(z-r)^3$ is a factor of $p(z)$. This is equivalent to the condition $p(r) = p'(r) = p''(r) = 0$. Differentiation of the KEY identity (3.5) twice with respect to z gives

$$\mathrm{L}\big(t^2 e^{zt}\big) = p(z)\,t^2 e^{zt} + 2p'(z)\,t\,e^{zt} + p''(z)\,e^{zt}\,.$$

Evaluating this at $z = r$ shows that

$$\mathrm{L}\big(t^2 e^{rt}\big) = p(r)\,t^2 e^{rt} + 2p'(r)\,t\,e^{rt} + p''(r)\,e^{rt} = 0\,.$$

Hence, e^{rt}, $t\,e^{rt}$, and t^2e^{rt} are solutions of the homogeneous equation $\mathrm{L}y = 0$. Because

$$\frac{d}{dt}e^{rt} = r\,e^{rt}\,, \qquad \frac{d}{dt}(t\,e^{rt}) = rt\,e^{rt} + e^{rt}\,, \qquad \frac{d}{dt}(t^2e^{rt}) = rt^2e^{rt} + 2t\,e^{rt}\,,$$

and

$$\frac{d^2}{dt^2}e^{rt} = r^2e^{rt}\,, \quad \frac{d^2}{dt^2}(t\,e^{rt}) = r^2t\,e^{rt} + 2r\,e^{rt}\,, \quad \frac{d^2}{dt^2}(t^2e^{rt}) = r^2t^2e^{rt} + 4rt\,e^{rt} + 2\,e^{rt}\,,$$

the Wronskian of these solutions is

$$\det\begin{pmatrix} e^{rt} & t\,e^{rt} & t^2e^{rt} \\ r\,e^{rt} & rt\,e^{rt} + e^{rt} & rt^2e^{rt} + 2t\,e^{rt} \\ r^2e^{rt} & r^2t\,e^{rt} + 2r\,e^{rt} & r^2t^2e^{rt} + 4rt\,e^{rt} + 2\,e^{rt} \end{pmatrix} = 2e^{3rt} \neq 0\,.$$

These solutions are therefore independent.

More generally, recall that r is a real root of $p(z)$ of multiplicity m when $(z-r)^m$ is a factor of $p(z)$. This is equivalent to the condition $p(r) = p'(r) = \cdots = p^{(m-1)}(r) = 0$. Differentiation of the KEY identity (3.5) k times with respect to z gives

$$\mathrm{L}\big(t^k e^{zt}\big) = p(z)\,t^k e^{zt} + kp'(z)\,t^{k-1}e^{zt} + \cdots + kp^{(k-1)}(z)\,t\,e^{zt} + p^{(k)}(z)\,e^{zt}\,.$$

Evaluating this at $z = r$ when $k = 0, 1, \cdots, m-1$ shows that

$$\mathrm{L}\big(t^k e^{rt}\big) = p(r)\,t^k e^{rt} + kp'(r)\,t^{k-1}e^{rt} + \cdots + kp^{(k-1)}(r)\,t\,e^{rt} + p^{(k)}(r)\,e^{rt} = 0\,.$$

Hence,

$$e^{rt}\,, \qquad t\,e^{rt}\,, \qquad \cdots \qquad t^{m-1}e^{rt}\,,$$

are m solutions of the homogeneous equation $\mathrm{L}y = 0$. These solutions are in fact independent, but we will not show that here.

Example: Find a general solution of

$$\mathrm{D}^6y - 5\mathrm{D}^5y + 6\mathrm{D}^4y + 4\mathrm{D}^3y - 8\mathrm{D}^2y = 0\,, \qquad \text{where} \quad \mathrm{D} = \frac{d}{dt}\,.$$

The characteristic polynomial is

$$p(z) = z^6 - 5z^5 + 6z^4 + 4z^3 - 8z^2 = z^2(z+1)(z-2)^3\,.$$

Its six roots (counting multiplicity) are 0, 0, -1, 2, 2, 2. The solutions associated with the double root 0 are 1 and t. The solution associated with the simple root -1 is e^{-t}. The solutions associated with the triple root 2 are e^{2t}, $t\,e^{2t}$, and t^2e^{2t}. A general solution is therefore

$$y = c_1 + c_2t + c_3e^{-t} + c_4e^{2t} + c_5t\,e^{2t} + c_6t^2e^{2t}\,.$$

3.3: Complex Roots of Characteristic Polynomials. Consider the problem of finding a general solution of

$$\mathrm{L}y = \mathrm{D}^2 y + 9y = 0\,, \qquad \text{where} \quad \mathrm{D} = \frac{d}{dt}\,.$$

The characteristic polynomial is $p(z) = z^2 + 9$, which clearly has no real roots. However, this can be factored over the complex numbers as

$$p(z) = (z - i3)(z + i3)\,,$$

where $i = \sqrt{-1}$. Its roots are the conjugate pair $i3$ and $-i3$. We claim that e^{i3t} and e^{-i3t} are independent complex-valued solutions of $\mathrm{L}y = 0$. We must first recall what is meant by such complex-valued solutions. We must then see how to generate real-valued solutions from them.

You should recall the Euler identity from your study of calculus. It states that for every real θ one has

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)\,.$$

This identity is the key to making sense of complex exponentials. In particular, for any real number s one has

$$e^{ist} = \cos(st) + i\sin(st)\,.$$

The derivative of this function with respect to t is then

$$\begin{aligned}\mathrm{D}e^{ist} &= \mathrm{D}\big(\cos(st) + i\sin(st)\big) = \mathrm{D}\cos(st) + i\mathrm{D}\sin(st)\\ &= -s\sin(st) + is\cos(st) = is\big(\cos(st) + i\sin(st)\big) = ise^{ist}\,.\end{aligned}$$

More generally, for any real numbers r and s one has

$$e^{(r+is)t} = e^{rt+ist} = e^{rt}e^{ist} = e^{rt}\big(\cos(st) + i\sin(st)\big)\,.$$

By the product rule and our result that $\mathrm{D}e^{ist} = ise^{ist}$, the derivative of this function with respect to t is

$$\begin{aligned}\mathrm{D}e^{(r+is)t} &= \mathrm{D}\big(e^{rt}e^{ist}\big) = \mathrm{D}\big(e^{rt}\big)e^{ist} + e^{rt}\mathrm{D}\big(e^{ist}\big)\\ &= re^{rt}e^{ist} + ise^{rt}e^{ist} = (r + is)e^{(r+is)t}\,.\end{aligned}$$

We thereby see that for any complex number z we have $\mathrm{D}e^{zt} = ze^{zt}$, from which it follows that for any polynomial $p(z)$ one has

$$p(\mathrm{D})e^{zt} = p(z)e^{zt}\,. \tag{3.6}$$

In particular, the KEY identity holds for any complex z!

Let $p(z)$ be the characteristic polynomial of the differential operator L given by (3.3). Because $p(z)$ has real coefficients, it has the property that

$$p(\overline{z}) = \overline{p(z)} \quad \text{for every complex } z\,,$$

where the bar denotes complex conjugate — i.e. $\overline{X + iY} = X - iY$ for any real X and Y. Thus, if $p(r + is) = 0$ then

$$p(r - is) = p(\overline{r + is}) = \overline{p(r + is)} = 0\,.$$

Hence, roots of $p(z)$ come in conjugate pairs; if $r + is$ is a root then so is $r - is$.

By the KEY identity (3.6), if $p(z)$ has a conjugate pair of roots $r + is$ and $r - is$ then $e^{(r+is)t}$ and $e^{(r-is)t}$ are a pair of complex-valued solutions of equation (3.1) — namely, they satisfy

$$\mathrm{L}e^{(r+is)t} = 0\,, \qquad \mathrm{L}e^{(r-is)t} = 0\,. \tag{3.7}$$

Because $e^{(r+is)t} = e^{rt}\cos(st) + ie^{rt}\sin(st)$, its real and imaginary parts are

$$\mathrm{Re}\big(e^{(r+is)t}\big) = e^{rt}\cos(st)\,, \qquad \mathrm{Im}\big(e^{(r+is)t}\big) = e^{rt}\sin(st)\,.$$

Recall that for any complex Z its real and imaginary parts can be expressed as

$$\mathrm{Re}(Z) = \frac{Z + \overline{Z}}{2}\,, \qquad \mathrm{Im}(Z) = \frac{Z - \overline{Z}}{i2}\,.$$

We therefore have

$$\begin{aligned} e^{rt}\cos(st) &= \mathrm{Re}\big(e^{(r+is)t}\big) = \frac{e^{(r+is)t} + e^{(r-is)t}}{2}\,, \\ e^{rt}\sin(st) &= \mathrm{Im}\big(e^{(r+is)t}\big) = \frac{e^{(r+is)t} - e^{(r-is)t}}{i2}\,. \end{aligned}$$

It then follows from (3.7) that

$$\begin{aligned} \mathrm{L}\big(e^{rt}\cos(st)\big) &= \mathrm{L}\left(\frac{e^{(r+is)t} + e^{(r-is)t}}{2}\right) = \frac{1}{2}\Big(\mathrm{L}e^{(r+is)t} + \mathrm{L}e^{(r-is)t}\Big) = 0\,, \\ \mathrm{L}\big(e^{rt}\sin(st)\big) &= \mathrm{L}\left(\frac{e^{(r+is)t} - e^{(r-is)t}}{i2}\right) = \frac{1}{i2}\Big(\mathrm{L}e^{(r+is)t} - \mathrm{L}e^{(r-is)t}\Big) = 0\,. \end{aligned}$$

In other words, when the characteristic polynomial $p(z)$ of the differential operator L has a conjugate pair of roots $r + is$ and $r - is$ then $e^{rt}\cos(st)$ and $e^{rt}\sin(st)$ are a pair of real-valued solutions of equation (3.1). You can easily check that they are linearly independent when $s \neq 0$.

If $p(z)$ has a conjugate pair of roots $r+is$ and $r-is$ with $s\neq 0$ then it has the pair of complex factors $(z-r-is)$ and $(z-r+is)$. Because

$$(z-r-is)(z-r+is)=(z-r)^2-(is)^2=(z-r)^2+s^2\,,$$

we see that $p(z)$ has the irreducible real factor $(z-r)^2+s^2$. Conversely, if $p(z)$ has the irreducible real factor $(z-r)^2+s^2$ then it has the conjugate pair of roots $r+is$ and $r-is$.

Example: Find a general solution of

$$\mathrm{L}y=\mathrm{D}^2y+2\mathrm{D}y+5y=0\,,\qquad \text{where}\quad \mathrm{D}=\frac{d}{dt}\,.$$

Solution: The characteristic polynomial is

$$p(z)=z^2+2z+5=(z+1)^2+4=(z+1)^2+2^2\,,$$

which has the conjugate pair of roots $-1+i2$ and $-1-i2$. A general solution is therefore

$$y=c_1e^{-t}\cos(2t)+c_2e^{-t}\sin(2t)\,.$$

Example: Find a general solution of

$$\mathrm{L}y=\mathrm{D}^2y+9y=0\,,\qquad \text{where}\quad \mathrm{D}=\frac{d}{dt}\,.$$

Solution: The characteristic polynomial is

$$p(z)=z^2+9=z^2+3^2\,,$$

which has the conjugate pair of roots $i3$ and $-i3$. A general solution is therefore

$$y=c_1\cos(3t)+c_2\sin(3t)\,.$$

Example: Find a general solution of

$$\mathrm{L}y=(\mathrm{D}+5)^3(\mathrm{D}^2+4\mathrm{D}+5)(\mathrm{D}^2+4)y=0\,,\qquad \text{where}\quad \mathrm{D}=\frac{d}{dt}\,.$$

Solution: The characteristic polynomial is

$$p(z)=(z+5)^3(z^2+4z+5)(z^2+4)=(z+5)^3\big((z+2)^2+1^2\big)\big(z^2+2^2\big)\,,$$

which has the real roots -5, -5, -5, the conjugate pair of roots $-2+i$, $-2-i$, and the conjugate pair of roots $i2$, $-i2$. A general solution is therefore

$$y=c_1e^{-5t}+c_2t\,e^{-5t}+c_3t^2e^{-5t}+c_4e^{-2t}\cos(t)+c_5e^{-2t}\sin(t)+c_6\cos(2t)+c_7\sin(2t)\,.$$

The fundamental theorem of algebra says that any polynomial $p(z)$ of degree n can be written as the product of n linear factors:

$$p(z) = (z - z_1)(z - z_2) \cdots (z - z_n) \,,$$

where z_1, z_2, $\cdots$, z_n are complex numbers that are roots of $p(z)$ — i.e. $p(z_j) = 0$ for each z_j. Conversely, if $p(r + is) = 0$ then $r + is = z_j$ for some z_j. We say $r + is$ is a root of $p(z)$ of multiplicity m if $r + is = z_j$ for m of the z_j. In other words, $r + is$ is a root of $p(z)$ of multiplicity m if $(z - r - is)^m$ is a factor of $p(z)$.

If all the coefficients of a polynomial $p(z)$ are real then it is called a *real polynomial*. Characteristic polynomials of linear differential operators in the form (3.3) are real polynomials. If $p(z)$ is a real polynomial and $r+is$ is a root of $p(z)$ of multiplicity m its conjugate $r - is$ is also a root of $p(z)$ of multiplicity m. If $s \neq 0$ then this means that $(z - r - is)^m$ and $(z - r + is)^m$ are distinct complex factors of $p(z)$, which means that $\big((z - r)^2 + s^2\big)^m$ is a real factor of $p(z)$. Conversely, if $\big((z - r)^2 + s^2\big)^m$ is a factor of $p(z)$ for some real r and s and some positive integer m and if $s \neq 0$ then $r + is$ and $r - is$ are distinct roots of $p(z)$ of multiplicity m.

Example: Find all the roots of $p(z) = (z^3 - 2z^2)(z^2 - 2z + 10)^3(z^2 + 4z + 29)$.

Solution: Because the degree of a factored polynomial is the sum of the degrees of its factors, you see that the degree of $p(z)$ is $3 + 6 + 2 = 11$. Because $p(z)$ has degree 11, it must have 11 roots counting multiplicities. Because

$$p(z) = z^2(z - 2)\big((z - 1)^2 + 3^2\big)^3\big((z + 2)^2 + 5^2\big) \,,$$

the 11 roots are 0, 0, 2, $1 \pm i3$, $1 \pm i3$, $1 \pm i3$, $-2 \pm i5$. Here each $r \pm is$ denotes two distinct roots. The real root 0 has multiplicity 2 while the complex roots $1 + i3$ and $1 - i3$ have multiplicity 3.

Now let $p(z)$ be the characteristic polynomial of an n^{th} order linear differential operator L in the form (3.3). We know that $p(z)$ has n complex roots counting multiplicities. We already know that if r is a real root of $p(z)$ of multiplicity m then $\mathrm{L}y = 0$ has the m linearly independent real solutions given by

$$e^{rt} \,, \qquad t\, e^{rt} \,, \qquad \cdots \qquad t^{m-1} e^{rt} \,. \tag{3.8}$$

Below we will show that if $r \pm is$ is a conjugate pair of roots of $p(z)$ of multiplicity m then $\mathrm{L}y = 0$ has the $2m$ solutions

$$\begin{aligned} &e^{rt}\cos(st) \,, \qquad t\, e^{rt}\cos(st) \,, \qquad \cdots \qquad t^{m-1}e^{rt}\cos(st) \,, \\ &e^{rt}\sin(st) \,, \qquad t\, e^{rt}\sin(st) \,, \qquad \cdots \qquad t^{m-1}e^{rt}\sin(st) \,. \end{aligned} \tag{3.9}$$

The n roots of $p(z)$ therefore generate n solutions of $\mathrm{L}y = 0$ by recipes (3.8) and (3.9). Moreover, it can be shown that these solutions are linearly independent, and thereby are a fundamental set of solutions for the problem.

Example: Find a general solution of

$$\mathrm{D}^4 y + 8\mathrm{D}^2 y + 16y = 0\,, \qquad \text{where} \quad \mathrm{D} = \frac{d}{dt}\,.$$

Solution: The characteristic polynomial is

$$p(z) = z^4 + 8z^2 + 16 = \big(z^2 + 4\big)^2 = \big(z^2 + 2^2\big)^2\,.$$

Its 4 roots are $\pm i2$, $\pm i2$. A general solution is therefore

$$y = c_1 \cos(2t) + c_2 \sin(2t) + c_3 t\, \cos(2t) + c_4 t\, \sin(2t)\,.$$

Example: Find a general solution of

$$\big(\mathrm{D}^3 - 2\mathrm{D}^2\big)\big(\mathrm{D}^2 - 2\mathrm{D} + 10\big)^3\big(\mathrm{D}^2 + 4\mathrm{D} + 29\big)y = 0\,, \qquad \text{where} \quad \mathrm{D} = \frac{d}{dt}\,.$$

Solution: The characteristic polynomial is

$$\begin{aligned} p(z) &= (z^3 - 2z^2)(z^2 - 2z + 10)^3(z^2 + 4z + 29) \\ &= z^2(z-2)\big((z-1)^2 + 3^2\big)^3\big((z+2)^2 + 5^2\big)\,. \end{aligned}$$

Its 11 roots are 0, 0, 2, $1 \pm i3$, $1 \pm i3$, $1 \pm i3$, $-2 \pm i5$. A general solution is therefore

$$\begin{aligned} y = c_1 + c_2 t + c_3 e^{2t} &+ c_4 e^t \cos(3t) + c_5 e^t \sin(3t) + c_6 t\, e^t \cos(3t) + c_7 t\, e^t \sin(3t) \\ &+ c_8 t^2 e^t \cos(3t) + c_9 t^2 e^t \sin(3t) + c_{10} e^{-2t} \cos(5t) + c_5 e^{-2t} \sin(5t)\,. \end{aligned}$$

Recall that recipe (3.8) was derived by evaluating the KEY identity and its first $m-1$ derivatives at the root $z = r$, and using that fact that $p(r) = p'(r) = \cdots = p^{(m-1)}(r) = 0$. Recipe (3.9) is derived in a similar way. If $r + is$ is a complex root $p(z)$ of multiplicity m then $(z - r - is)^m$ is a factor of $p(z)$. One can differentiate polynomials and e^{zt} with respect to the complex variable z exactly as if it were a real variable. Because $(z - r - is)^m$ is a factor of $p(z)$, you can show that

$$p(r + is) = p'(r + is) = \cdots = p^{(m-1)}(r + is) = 0\,.$$

Differentiation of the KEY identity (3.6) k times with respect to z gives

$$\mathrm{L}\big(t^k e^{zt}\big) = p(z)\, t^k e^{zt} + kp'(z)\, t^{k-1} e^{zt} + \cdots + kp^{(k-1)}(z)\, t\, e^{zt} + p^{(k)}(z)\, e^{zt}\,.$$

Evaluating this at $z = r + is$ when $k = 0, 1, \cdots, m-1$ shows that

$$\begin{aligned} \mathrm{L}\big(t^k e^{(r+is)t}\big) &= p(r+is)\, t^k e^{(r+is)t} + kp'(r+is)\, t^{k-1} e^{(r+is)t} + \cdots \\ &\qquad \cdots + kp^{(k-1)}(r+is)\, t\, e^{(r+is)t} + p^{(k)}(r+is)\, e^{(r+is)t} = 0\,. \end{aligned}$$

Similarly you can show that $\mathrm{L}\big(t^k e^{(r-is)t}\big) = 0$. Recipe (3.9) then follows by the taking real and imagingary parts of these complex-valued solutions.

Appendix A. Linear Algebraic Systems and Determinants

A.1: Linear Algebraic Systems. In all of the examples in section 2.1 we were able to find a unique solution c_1, c_2, $\cdots$, c_n to the linear algebraic system (2.2) which enabled us to solve the general initial value problem for any choice of initial data y_0, y_1, $\cdots$, y_{n-1}. In this section we will characterize when a linear algebraic system always has a unique solution.

A linear algebric system of n equations for n unknowns x_1, x_2, $\cdots$, x_n has the general form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \,, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \,, \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \,. \end{aligned} \tag{A.1}$$

Here the n^2 numbers $\{a_{ij} : i, j = 1, 2, \cdots, n\}$ are called the *coefficients* of the system and the n numbers b_1, b_2, $\cdots$, b_n are called the *forcing*. The system is called *homogeneous* if $b_1 = b_2 = \cdots = b_n = 0$, and *nonhomogenous* otherwise.

There are two questions regarding the existence of solutions that we want to address. The first is:

When does system (A.1) have a unique solution for every forcing b_1, b_2, $\cdots$, b_n*?*

The second is:

When does system (A.1) with $b_1 = b_2 = \cdots = b_n = 0$ *have a nonzero solution?*

Here "nonzero solution" means a solution with $x_k \neq 0$ for some index k — i.e. a solution that is not the "trivial solution" $x_1 = x_2 = \cdots = x_n = 0$.

These questions are clearly related. Let us suppose that system (A.1) has two different solutions for some set of numbers b_1, b_2, $\cdots$, b_n. We denote one of these solutions by x_1, x_2, $\cdots$, x_n and the other by y_1, y_2, $\cdots$, y_n. Set

$$z_1 = x_1 - y_1 \,, \qquad z_2 = x_2 - y_2 \,, \qquad \cdots \,, \qquad z_n = x_n - y_n \,.$$

Then one can show that z_1, z_2, $\cdots$, z_n is a nonzero solution of

$$\begin{aligned} a_{11}z_1 + a_{12}z_2 + \cdots + a_{1n}z_n &= 0 \,, \\ a_{21}z_1 + a_{22}z_2 + \cdots + a_{2n}z_n &= 0 \,, \\ &\vdots \\ a_{n1}z_1 + a_{n2}z_2 + \cdots + a_{nn}z_n &= 0 \,. \end{aligned} \tag{A.2}$$

But this is system (A.1) with $b_1 = b_2 = \cdots = b_n = 0$.

Conversely, if system (A.2) has a nonzero solution z_1, z_2, $\cdots$, z_n then no solution of (A.1) is unique for any forcing b_1, b_2, $\cdots$, b_n. To see this, let y_1, y_2, $\cdots$, y_n be a solution of (A.1) for some forcing b_1, b_2, $\cdots$, b_n:

$$\begin{aligned} a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n &= b_1 \,, \\ a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n &= b_2 \,, \\ &\vdots \\ a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n &= b_n \,. \end{aligned}$$

For any nonzero real number α set

$$x_1 = y_1 + \alpha z_1 \,, \qquad x_2 = y_2 + \alpha z_2 \,, \qquad \cdots \,, \qquad x_n = y_n + \alpha z_n \,.$$

Then one can show that x_1, x_2, $\cdots$, x_n is a solution of system (A.1) for the same forcing b_1, b_2, $\cdots$, b_n that is different than y_1, y_2, $\cdots$, y_n. Hence, the existence of a nonzero solution z_1, z_2, $\cdots$, z_n of (A.2) implies that for any given forcing b_1, b_2, $\cdots$, b_n, system (A.1) either has no solution or has many solutions. It therefore does not have a unique solution for any forcing.

A.2: Determinants. Answers to our questions can depend only on the coefficients $\{a_{ij} : i, j = 1, 2, \cdots, n\}$. It is helpful to write these coefficients as an $n\times n$ matrix A:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} .$$

The answers will be given in terms of a quantity $\det(A)$, called the *determinant* of A. For $n = 1$, 2, and 3 the quantity $\det(A)$ is given by

$$\begin{aligned} \det(\, a_{11} \,) &= a_{11} \,, \\ \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} &= a_{11}a_{22} - a_{12}a_{21} \,, \\ \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \,. \end{aligned} \tag{A.3}$$

The best way to remember these formulas is visually. The formula for the 2×2 determinant can be remembered as the product of the terms on the $\searrow$ diagonal minus the product of the terms on the $\swarrow$ diagonal. (Draw these two diagonal arrows on the above 2×2 matrix as was done in class.) The formula for the 3×3 determinant can be remembered by first augmenting the matrix by repeating the first two columns, thereby creating the 3×5 augmented matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{pmatrix} .$$

The formula is then the sum of the products of the terms on the ↘↘↘ diagonals minus the sum of the products of the terms on the ↙↙↙ diagonals. (Draw these six diagonal arrows on the above 3×5 augmented matrix as was done in class.) You can also use the "spaghetti" drawing on the 3×3 matrix.

In general the determinant of the $n\times n$ matrix A

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

can be expanded in terms of the determinants of $(n-1)\times(n-1)$ submatrices of A. Let A_{jk} denotes the $(n-1)\times(n-1)$ submatrix of A obtained by crossing out the j^{th} row and k^{th} column of A. Then for any j we can expand $\det(A)$ about the j^{th} row of A as

$$\det(A) = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} \det(A_{jk}) , \tag{A.4}$$

while for any k we can expand $\det(A)$ about the k^{th} column of A as

$$\det(A) = \sum_{j=1}^{n} (-1)^{j+k} a_{jk} \det(A_{jk}) , \tag{A.5}$$

These are called *Laplace formulas* for the determinant. Typically, one uses either (A.4) with $j = 1$ or (A.5) with $k = 1$. For example, for $n = 2$ and $j = 1$ formula (A.4) gives

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}\det(a_{22}) - a_{12}\det(a_{21}) = a_{11}a_{22} - a_{12}a_{21} .$$

For $n = 3$ and $j = 1$ formula (A.4) gives

$$\begin{aligned} \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} &= a_{11}\det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12}\det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13}\det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \\ &= a_{11}\big(a_{22}a_{33} - a_{23}a_{32}\big) - a_{12}\big(a_{21}a_{33} - a_{23}a_{31}\big) \\ &\qquad + a_{13}\big(a_{21}a_{32} - a_{22}a_{31}\big) \\ &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} \\ &\qquad + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} . \end{aligned}$$

For $n = 4$ and $j = 1$ formula (A.4) gives

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = a_{11}\det\begin{pmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{pmatrix} - a_{12}\det\begin{pmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{pmatrix}$$
$$+ a_{13}\det\begin{pmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{pmatrix} - a_{14}\det\begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}.$$

Each of the 3×3 deteninants above can be expanded further. The resulting formula for the determinant of a 4×4 matrix is the sum of 24 products. Fully expanded, a similar formula for the determinant of an $n{\times}n$ matrix is the sum of $n!$ products. There is no simple "diagonal" picture that can be used to remember these formulas visually when $n > 3$. However the Laplace formulas (A.4) and (A.5) allow you to compute determinants without difficulty provided that either n is not too large or A has a simple structure.

Exercise: Prove the following evaluation of the determinant of a triangular $n{\times}n$ matrix

$$\det\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_{n1} & \cdots & a_{n(n-1)} & a_{nn} \end{pmatrix} = a_{11}a_{22}\cdots a_{nn}\,.$$

Hint: Use the Laplace formula (A.4) and induction on n.

A.3: Existence of Solutions. The answers to our questions are as follows.

> **Theorem A.1:** System (A.1) has a unique solution for every forcing b_1, b_2, $\cdots$, b_n if and only if $\det(A) \neq 0$.

> **Theorem A.2:** System (A.2) has a nonzero solution if and only if $\det(A) = 0$.

Remark: Half of Theorem A.1 is implied by Theorem A.2. Indeed, Theorem A.2 implies that if $\det(A) = 0$ then system (A.2) has a nonzero solution. As we showed earlier, the existence of such a solution implies that for any given forcing b_1, b_2, $\cdots$, b_n, system (A.1) either has no solution or has many solutions. It therefore does not have a unique solution for any forcing. Hence, once Theorem A.2 is established, all that one needs to show to establish Theorem A.1 is that if $\det(A) \neq 0$ then system (A.1) has a unique solution.

We will not give proofs of Theorem A.1 and Theorem A.2 for general n because they are beyond the scope of this course. They are covered in sufficiently advanced linear algebra courses. However, we will give proofs of these theorems for the cases $n = 1$ and $n = 2$. While you will not be expected to know these proofs, you will be expected to know both theorems.

Proofs: When $n = 1$ system (A.2) is simply the single equation

$$a_{11}z_1 = 0\,. \tag{A.6}$$

Clearly, if $\det(A) = a_{11} \neq 0$ then $z_1 = 0$ is the only solution of (A.6). Conversely, if $\det(A) = a_{11} = 0$ then every z_1 satisfies (A.6). Hence, Theorem A.2 holds for $n = 1$.

When $n = 1$ system (A.1) is simply the single equation

$$a_{11}x_1 = b_1\,.$$

If $\det(A) = a_{11} \neq 0$ then this clearly has the unique solution

$$x_1 = \frac{b_1}{a_{11}}\,.$$

Hence, Theorem A.1 holds for $n = 1$.

When $n = 2$ system (A.2) is the two equations

$$\begin{aligned} a_{11}z_1 + a_{12}z_2 &= 0\,,\\ a_{21}z_1 + a_{22}z_2 &= 0\,. \end{aligned} \tag{A.7}$$

First eliminate z_2 by multiplying the first equation in (A.7) by a_{22}, the second by a_{12}, and then subtracting the results to obtain

$$\big(a_{11}a_{22} - a_{12}a_{21}\big)z_1 = 0\,. \tag{A.8a}$$

Similarly, eliminate z_1 by multiplying the second equation in (A.7) by a_{11}, the second by a_{21}, and then subtracting the results to obtain

$$\big(a_{11}a_{22} - a_{12}a_{21}\big)z_2 = 0\,. \tag{A.8b}$$

It is clear from (A.8) that if $\det(A) = a_{11}a_{22} - a_{12}a_{21} \neq 0$ then $z_1 = z_2 = 0$ is the only solution of (A.7). Conversely, if $\det(A) = a_{11}a_{22} - a_{12}a_{21} = 0$ then both

$$z_1 = a_{22}\,, \quad z_2 = -a_{21}\,, \qquad \text{and} \qquad z_1 = -a_{12}\,, \quad z_2 = a_{11}\,,$$

give solutions of (A.7), at least one of which will be nonzero unless $a_{11} = a_{12} = a_{12} = a_{12} = 0$. However, when $a_{11} = a_{12} = a_{12} = a_{12} = 0$ then any values of z_1 and z_2 satisfy (A.7). Hence, (A.7) has a nonzero solution in either case. Therefore, Theorem A.2 holds for $n = 2$.

When $n = 2$ system (A.1) is the two equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1\,,\\ a_{21}x_1 + a_{22}x_2 &= b_2\,. \end{aligned} \tag{A.9}$$

First eliminate x_2 by multiplying the first equation in (A.9) by a_{22}, the second by a_{12}, and then subtracting the results to obtain

$$\big(a_{11}a_{22} - a_{12}a_{21}\big)x_1 = b_1 a_{22} - a_{12}b_2\,. \tag{A.10a}$$

Similarly, eliminate x_1 by multiplying the second equation in (A.9) by a_{11}, the second by a_{21}, and then subtracting the results to obtain

$$\big(a_{11}a_{22} - a_{12}a_{21}\big)x_2 = a_{11}b_2 - b_1 a_{21}\,. \tag{A.10b}$$

It is clear from (A.10) that if $\det(A) = a_{11}a_{22} - a_{12}a_{21} \neq 0$ then for any choice of b_1 and b_2 the system (A.9) has the unique solution

$$x_1 = \frac{b_1 a_{22} - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}\,, \qquad x_2 = \frac{a_{11}b_2 - b_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}}\,.$$

Hence, Theorem A.1 holds for $n = 2$. $\square$