HIGHER-ORDER LINEAR
ORDINARY DIFFERENTIAL EQUATIONS III:
Laplace Transform Method

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Because the presentation of this material in class will differ from that in the book, I felt that notes that closely follow the class presentation might be appreciated.

6. Laplace Transform Method

The Laplace transform will allow us to transform an initial-value problem for a constant coefficient linear ordinary differential equation into a linear algebaric equation that can be easily solved. The solution of an initial-value problem can then be obtained from the solution of the algebaric equation by taking its so-called inverse Laplace transform.

6.1: Definition of the Transform. The Laplace transform of a function $f(t)$ defined over $t \geq 0$ is another function $\mathcal{L}[f](s)$ that is formally defined by

$$\mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t)\, \mathrm{d}t\,. \tag{6.1}$$

You should recall from calculus that the above definite integral is improper because its upper endpoint is ∞. The proper definition of the Laplace transform is therefore

$$\mathcal{L}[f](s) = \lim_{T\to\infty} \int_0^T e^{-st} f(t)\, \mathrm{d}t\,, \tag{6.2}$$

provided that the definite integrals over $[0, T]$ appearing in the above limit are proper. The Laplace transform $\mathcal{L}[f](s)$ is defined only at those s for which the limit in (6.2) exists.

Example. Use definition (6.2) to compute $\mathcal{L}[e^{at}](s)$ for any real a. From (6.2) you see that for any $s \neq a$ one has

$$\begin{aligned}\mathcal{L}[e^{at}](s) &= \lim_{T\to\infty} \int_0^T e^{-st} e^{at}\, \mathrm{d}t = \lim_{T\to\infty} \int_0^T e^{(a-s)t}\, \mathrm{d}t \\ &= \lim_{T\to\infty} \frac{e^{(a-s)t}}{a-s}\bigg|_{t=0}^{T} = \lim_{T\to\infty}\left[\frac{1}{s-a} - \frac{e^{(a-s)T}}{s-a}\right] = \begin{cases} \dfrac{1}{s-a} & \text{for } s > a\,, \\ \infty & \text{for } s < a\,, \end{cases}\end{aligned}$$

while for $s = a$ one has

$$\mathcal{L}[e^{at}](s) = \lim_{T\to\infty} \int_0^T e^{-(s-a)t}\, \mathrm{d}t = \lim_{T\to\infty} \int_0^T \mathrm{d}t = \lim_{T\to\infty} T = \infty\,.$$

Therefore $\mathcal{L}[e^{at}](s)$ is only defined for $s > a$ with

$$\mathcal{L}[e^{at}](s) = \frac{1}{s-a} \quad \text{for } s > a\,.$$

Example. Use definition (6.2) to compute $\mathcal{L}[t\,e^{at}](s)$ for any real a. From (6.2) you see that for any $s \neq a$ one has

$$
\begin{aligned}
\mathcal{L}[e^{at}](s) &= \lim_{T\to\infty} \int_0^T t\,e^{-st} e^{at}\,\mathrm{d}t = \lim_{T\to\infty} \int_0^T t\,e^{(a-s)t}\,\mathrm{d}t \\
&= \lim_{T\to\infty} \left(\frac{t}{a-s} - \frac{1}{(a-s)^2} \right) e^{(a-s)t} \bigg|_{t=0}^{T} \\
&= \lim_{T\to\infty} \left[\frac{1}{(s-a)^2} - \left(\frac{T}{s-a} + \frac{1}{(s-a)^2} \right) e^{(a-s)T} \right] = \begin{cases} \dfrac{1}{(s-a)^2} & \text{for } s > a\,, \\ \infty & \text{for } s < a\,, \end{cases}
\end{aligned}
$$

while for $s = a$ one has

$$
\mathcal{L}[e^{at}](s) = \lim_{T\to\infty} \int_0^T t\,e^{-(s-a)t}\,\mathrm{d}t = \lim_{T\to\infty} \int_0^T t\,\mathrm{d}t = \lim_{T\to\infty} \tfrac{1}{2}T^2 = \infty\,.
$$

Therefore $\mathcal{L}[t\,e^{at}](s)$ is only defined for $s > a$ with

$$
\mathcal{L}[t\,e^{at}](s) = \frac{1}{(s-a)^2} \quad \text{for } s > a\,.
$$

Example. Use definition (6.2) to compute $\mathcal{L}[e^{ibt}](s)$ for any real $b \neq 0$. From (6.2) you see that for any real s one has

$$
\begin{aligned}
\mathcal{L}[e^{ibt}](s) &= \lim_{T\to\infty} \int_0^T e^{-st} e^{ibt}\,\mathrm{d}t = \lim_{T\to\infty} \int_0^T e^{-(s-ib)t}\,\mathrm{d}t = \lim_{T\to\infty} \left(-\frac{e^{-(s-ib)t}}{s-ib} \right) \bigg|_{t=0}^{T} \\
&= \lim_{T\to\infty} \left[\frac{1}{s-ib} - \frac{e^{-(s-ib)T}}{s-ib} \right] = \begin{cases} \dfrac{1}{s-ib} & \text{for } s > 0\,, \\ \text{undefined} & \text{for } s \leq 0\,. \end{cases}
\end{aligned}
$$

Therefore $\mathcal{L}[e^{ibt}](s)$ is only defined for $s > 0$ with

$$
\mathcal{L}[e^{ibt}](s) = \frac{1}{s-ib} \quad \text{for } s > 0\,.
$$

6.2: Properties of the Transform. If we always had to return to the definition of the Laplace transform everytime we wanted to apply it, it would not be easy to use. Rather, we will use the definition to compute the Laplace transform for a few basic functions and to establish some general properties that will allow us to build formulas for more complicated functions. The most important such property is the fact that the Laplace transform $\mathcal{L}$ is a linear operator.

Theorem. If $\mathcal{L}[f](s)$ and $\mathcal{L}[g](s)$ exist for some s then so does $\mathcal{L}[f+g](s)$ and $\mathcal{L}[cf](s)$ for every constant c with

$$\mathcal{L}[f+g](s) = \mathcal{L}[f](s) + \mathcal{L}[g](s)\,, \qquad \mathcal{L}[cf](s) = c\mathcal{L}[f](s)\,. \tag{6.3}$$

Proof. This follows directly from definition (6.2) and the facts that definite integrals and limits depend linearly on their arguments. Specifically, one sees that

$$\begin{aligned}
\mathcal{L}[f+g](s) &= \lim_{T\to\infty} \int_0^T e^{-st}\big(f(t)+g(t)\big)\,\mathrm{d}t \\
&= \lim_{T\to\infty} \int_0^T e^{-st} f(t)\,\mathrm{d}t + \lim_{T\to\infty} \int_0^T e^{-st} g(t)\,\mathrm{d}t = \mathcal{L}[f](s) + \mathcal{L}[g](s)\,, \\
\mathcal{L}[cf](s) &= \lim_{T\to\infty} \int_0^T e^{-st} cf(t)\,\mathrm{d}t = c \lim_{T\to\infty} \int_0^T e^{-st} f(t)\,\mathrm{d}t = c\mathcal{L}[f](s)\,.
\end{aligned}$$

□

Example. Compute $\mathcal{L}[\cos(bt)](s)$ and $\mathcal{L}[\sin(bt)](s)$ for any real $b \neq 0$. This can be done by using the Euler identity $e^{ibt} = \cos(bt) + i\sin(bt)$ and the linearity of $\mathcal{L}$. Then

$$\mathcal{L}[\cos(bt)](s) + i\mathcal{L}[\sin(bt)](s) = \mathcal{L}[e^{ibt}](s) = \frac{1}{s-ib} = \frac{s+ib}{s^2+b^2} \quad \text{for } s > 0\,.$$

It follows that

$$\begin{aligned}
\mathcal{L}[\cos(bt)](s) &= \frac{s}{s^2+b^2} \quad \text{for } s > 0\,, \\
\mathcal{L}[\sin(bt)](s) &= \frac{b}{s^2+b^2} \quad \text{for } s > 0\,.
\end{aligned}$$

Another general property of the Laplace transform is that it turns multiplication by an exponential in t into a translation of s.

Theorem. If $\mathcal{L}[f](s)$ exists for every $s > \alpha$ and a is any real number then $\mathcal{L}[e^{at}f(t)](s)$ exists for every $s > \alpha + a$ with

$$\mathcal{L}[e^{at}f(t)](s) = \mathcal{L}[f](s-a)\,.$$

Proof. This follows directly from definition (6.2). Specifically, one sees that

$$\mathcal{L}[e^{at}f](s) = \lim_{T\to\infty} \int_0^T e^{-st} e^{at} f(t)\,\mathrm{d}t = \lim_{T\to\infty} \int_0^T e^{-(s-a)t} f(t)\,\mathrm{d}t = \mathcal{L}[f](s-a)\,.$$

□

Examples. From our previous examples and the above theorem we see that

$$\begin{aligned} \mathcal{L}[e^{(a+ib)t}](s) &= \frac{1}{s-a-ib} \qquad \text{for } s > a\,, \\ \mathcal{L}[e^{at}\cos(bt)](s) &= \frac{s-a}{(s-a)^2+b^2} \quad \text{for } s > a\,, \\ \mathcal{L}[e^{at}\sin(bt)](s) &= \frac{b}{(s-a)^2+b^2} \quad \text{for } s > a\,. \end{aligned}$$

The Laplace transform also turns a translation of t into multiplication by an exponential in s. Notice that $\mathcal{L}[f](s)$ only depends on the values of $f(t)$ over $[0.\infty)$. Therefore before we translate f we multiply it by the *unit step function* $u(t)$ defined by

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0\,, \\ 0 & \text{for } t < 0\,. \end{cases} \tag{6.4}$$

Because the functions uf and f agree over $[0,\infty)$, it is clear that $\mathcal{L}[uf](s) = \mathcal{L}[f](s)$. We now consider the Laplace transform of the translation $u(t-c)f(t-c)$ for every $c > 0$.

Theorem. If $\mathcal{L}[f](s)$ exists for every $s > \alpha$ and $c > 0$ then $\mathcal{L}[u(t-c)f(t-c)](s)$ exists for every $s > \alpha$ with

$$\mathcal{L}[u(t-c)f(t-c)](s) = e^{-cs}\mathcal{L}[f](s) \quad \text{for } s > \alpha\,.$$

Proof. For every $T > c$ one has

$$\begin{aligned} \int_0^T e^{-st}u(t-c)f(t-c)\,\mathrm{d}t &= \int_c^T e^{-st}f(t-c)\,\mathrm{d}t = e^{-cs}\int_c^T e^{-s(t-c)}f(t-c)\,\mathrm{d}t \\ &= e^{-cs}\int_0^{T-c} e^{-st'}f(t')\,\mathrm{d}t'\,. \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{L}[u(t-c)f(t-c)](s) &= \lim_{T\to\infty}\int_0^T e^{-st}u(t-c)f(tc)\,\mathrm{d}t \\ &= e^{-cs}\lim_{T\to\infty}\int_0^{T-c} e^{-st}f(t)\,\mathrm{d}t = e^{-cs}\mathcal{L}[f](s) \quad \text{for } s > \alpha\,. \end{aligned}$$

□

6.3: Existence and Differentiablity of the Transform. In each of the above examples the definite integrals over $[0, T]$ that appear in the limit (6.2) were proper. Indeed, we were able to evaluate the definite integrals analytically and determine the limit (6.2) for every real s. More generally, from calculus we know that a definite integral over $[0, T]$ is proper whenever its integrand is:

- bounded over $[0, T]$,
- continuous at all but a finite number of points in $[0, T]$.

Such an integrand is said to be *piecewise continuous* over $[0, T]$. Because for e^{-st} is continuous (and therefore bounded) function of t over every $[0, T]$ for each real s, the definite integrals over $[0, T]$ that appear in the limit (6.2) will be proper whenever $f(t)$ is *piecewise continuous* over every $[0, T]$.

Example. The function

$$f(t) = \begin{cases} 0 & \text{for } 0 \leq t < \pi\,, \\ \cos(t) & \text{for } t \geq \pi\,, \end{cases}$$

is piecewise continuous over every $[0, T]$ because it is clearly bounded over $[0, \infty)$ and its only discontinuity is at the point $t = \pi$.

Example. The so-called *sawtooth* function

$$f(t) = t - k \quad \text{for } k \leq t < k + 1 \text{ where } k = 0, 1, 2, 3, \cdots\,,$$

is piecewise continuous over every $[0, T]$ because it is clearly bounded over $[0, \infty)$ and has discontinuities at the points $t = 1, 2, 3, \cdots$, only a finite number of which lie in each $[0, T]$.

If we assume that $f(t)$ is piecewise continuous over every $[0, T]$, we still have to give a condition under which the limit (6.2) will exist for certain s. Such a condition is provided by the following definition.

Definition. A function $f(t)$ defined over $[0, \infty)$ is said to be of *exponential order α as $t \to \infty$* provided that for every $\sigma > \alpha$ there exist K_σ and T_σ such that

$$|f(t)| \leq K_\sigma e^{\sigma t} \quad \text{for every } t \geq T_\sigma\,. \tag{6.5}$$

Roughly speaking, a function is of exponential order α as $t \to \infty$ if its absolute value does not grow faster than $e^{\sigma t}$ as $t \to \infty$ for every $\sigma > \alpha$.

Example. The function e^{at} is of exponential order a as $t \to \infty$ because (6.5) holds with $K_\sigma = 1$ and $T_\sigma = 0$ for every $\sigma > a$.

Example. The function $\cos(bt)$ is of exponential order 0 as $t \to \infty$ because (6.5) holds with $K_\sigma = 1$ and $T_\sigma = 0$ for every $\sigma > 0$.

Example. For every $p > 0$ the function t^p is of exponential order 0 as $t \to \infty$. Indeed, for every $\sigma > 0$ the function $e^{-\sigma t} t^p$ takes on its maximum over $[0, \infty)$ at $t = p/\sigma$, whereby

$$e^{-\sigma t} t^p \le \left(\frac{p}{e\sigma}\right)^p \quad \text{for every } t \ge 0\,.$$

It follows that (6.5) holds with $K_\sigma = (e\sigma/p)^{-p}$ and $T_\sigma = 0$ for every $\sigma > 0$.

One can show that if functions f and g are of exponential orders α and β respectively as $t \to \infty$ then the function $f + g$ is of exponential order $\max\{\alpha, \beta\}$ as $t \to \infty$, while the function fg is of exponential order $\alpha + \beta$ as $t \to \infty$.

Example. For every real a the function $e^{at} + e^{-at}$ is of exponential order $|a|$ as $t \to \infty$. This is because the functions e^{at} and e^{-at} are exponential orders a and $-a$ respectively as $t \to \infty$, and because $|a| = \max\{a, -a\}$.

Example. For every $p > 0$ and every real a and b the function $t^p e^{at} \cos(bt)$ is of exponential order a as $t \to \infty$. This is because the functions t^p, e^{at}, and $\cos(bt)$ are of exponential orders 0, a, and 0 respectively as $t \to \infty$.

The fact you should know about the existence of the Laplace transform for certain s is the following.

Theorem. Let $f(t)$ be

- piecewise continuous over every $[0, T]$,
- of exponential order α as $t \to \infty$.

Then for every positive integer k the function $t^k f(t)$ has these same properties. The function $F(s) = \mathcal{L}[f](s)$ is defined for every $s > \alpha$. Moreover, $F(s)$ is infinitely differentiable over $s > \alpha$ with

$$\mathcal{L}[t^k f(t)](s) = (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}s^k} F(s) \quad \text{for } s > \alpha\,. \tag{6.6}$$

Proof. Formula (6.6) can be derived formally

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st} f(t)\,\mathrm{d}t\,,\\
\frac{\mathrm{d}}{\mathrm{d}s} F(s) &= -\int_0^\infty t\,e^{-st} f(t)\,\mathrm{d}t\,,\\
\frac{\mathrm{d}^2}{\mathrm{d}s^2} F(s) &= \int_0^\infty t^2 e^{-st} f(t)\,\mathrm{d}t\,,\\
&\vdots\\
\frac{\mathrm{d}^k}{\mathrm{d}s^k} F(s) &= (-1)^k \int_0^\infty t^k e^{-st} f(t)\,\mathrm{d}t\,.
\end{aligned}$$

A correct proof would require some justification of taking the derivatives inside the above imporper integrals. We will not go into those details here. We will however give an easier proof of the fact that $F(s)$ is defined for $s > \alpha$. The proof uses the direct comparison test for the convergence of improper integrals. That test implies that if $g(t)$ and $G(t)$ are piecewise continuous over every $[0, T]$ such that $|g(t)| \leq G(t)$ for every $t \geq 0$ then

$$\int_0^\infty G(t)\,\mathrm{d}t \quad \text{converges} \qquad \Longrightarrow \qquad \int_0^\infty g(t)\,\mathrm{d}t \quad \text{converges}\,.$$

Let $s > \alpha$ and apply this test to $g(t) = e^{-st}f(t)$. Pick σ so that $\alpha < \sigma < s$. Because $f(t)$ is of exponential order α as $t \to \infty$ and $\sigma > \alpha$ there exist K_σ and T_σ such that (6.5) holds. Because $g(t) = e^{-st}f(t)$ is bounded over $[0, T_\sigma]$ there exists B_σ such that $|g(t)| \leq B_\sigma$ over $[0, T_\sigma]$. It thereby follows that

$$|g(t)| = e^{-st}|f(t)| \leq G(t) \equiv \begin{cases} B_\sigma & \text{for } 0 \leq t < T_\sigma \\ K_\sigma e^{(\sigma - s)t} & \text{for } t \geq T_\sigma\,. \end{cases}$$

Because $s > \sigma$ for this $G(t)$ you can show that

$$\int_0^\infty G(t)\,\mathrm{d}t = \lim_{T \to \infty} \int_0^T G(t)\,\mathrm{d}t \quad \text{converges}\,.$$

It follows that the limit in (6.2) converges, whereby $F(s) = \mathcal{L}[f](s)$ is defined at s. $\square$

Example. Because

$$\mathcal{L}[e^{(a+ib)t}](s) = \frac{1}{s - a - ib} \quad \text{for } s > a\,,$$

for every nonnegative integer k it follows from the above theorem that

$$\mathcal{L}[t^k e^{(a+ib)t}](s) = (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}s^k} \frac{1}{s - a - ib} = \frac{n!}{(s - a - ib)^{n+1}} \quad \text{for } s > a\,.$$

In particular, for every nonnegative integer k

$$\mathcal{L}[t^k](s) = \frac{k!}{s^{k+1}} \quad \text{for } s > 0\,,$$

$$\mathcal{L}[t^k e^{at}](s) = \frac{k!}{(s - a)^{k+1}} \quad \text{for } s > a\,.$$

$$\mathcal{L}[t^k e^{at}\cos(bt)](s) = \mathrm{Re}\biggl(\frac{k!}{(s - a - ib)^{k+1}}\biggr) \quad \text{for } s > a\,,$$

$$\mathcal{L}[t^k e^{at}\sin(bt)](s) = \mathrm{Im}\biggl(\frac{k!}{(s - a - ib)^{k+1}}\biggr) \quad \text{for } s > a\,.$$

6.4: Transform of Dervatives and Initial-Value Problems. The previous result shows that the Laplace transform turns a multiplication by t into a derivative with respect to s. The next result shows the Laplace transform turns a derivative with respect to t into a multiplication by s. This is why the Laplace transform can be used to transform initial-value problems into algebraic problems.

Theorem. Let $f(t)$ be continuous over $[0,\infty)$ such that

- $f(t)$ is of exponential order α as $t \to \infty$,
- $f'(t)$ is piecewise continuous over every $[0,T]$.

Then $\mathcal{L}[f'](s)$ is defined for every $s > \alpha$ with

$$\mathcal{L}[f'](s) = s\,\mathcal{L}[f](s) - f(0)\,.$$

Proof. Let $s > \alpha$. By definition (6.2), an integration by parts, the fact that $f(t)$ is of exponential order α as $t \to \infty$, and the fact that $\mathcal{L}[f](s)$ exists, one sees that

$$\begin{aligned}
\mathcal{L}[f'](s) &= \lim_{T\to\infty} \int_0^T e^{-st} f'(t)\,\mathrm{d}t = \lim_{T\to\infty} \left[e^{-st} f(t)\Big|_{t=0}^{T} + s \int_0^T e^{-st} f(t)\,\mathrm{d}t \right] \\
&= \lim_{T\to\infty} e^{-sT} f(T) - f(0) + s \lim_{T\to\infty} \int_0^T e^{-st} f(t)\,\mathrm{d}t = -f(0) + s\,\mathcal{L}[f](s)\,.
\end{aligned}$$

□

If $f(t)$ is sufficiently differentiable then the previous result can be applied repeatedly. For example, if $f(t)$ is twice differentiable then

$$\begin{aligned}
\mathcal{L}[f''](s) &= s\,\mathcal{L}[f'](s) - f'(0) = s\big(s\,\mathcal{L}[f](s) - f(0)\big) - f'(0) \\
&= s^2\mathcal{L}[f](s) - s\,f(0) - f'(0)\,.
\end{aligned}$$

If $f(t)$ is thrice differentiable then

$$\begin{aligned}
\mathcal{L}[f'''](s) &= s\,\mathcal{L}[f''](s) - f''(0) = s\big(s^2\mathcal{L}[f](s) - s\,f(0) - f'(0)\big) - f''(0) \\
&= s^3\mathcal{L}[f](s) - s^2 f(0) - s\,f'(0) - f''(0)\,.
\end{aligned}$$

Proceeding in this way you can use induction to prove the following.

Theorem. Let $f(t)$ be n-times differentiable over $[0,\infty)$ such that

- $f(t)$, $f'(t)$, $\cdots$, $f^{(n-1)}(t)$ are of exponential order α as $t \to \infty$,
- $f^{(n)}(t)$ is piecewise continuous over every $[0,T]$.

Then $\mathcal{L}[f^{(n)}](s)$ is defined for every $s > \alpha$ with

$$\mathcal{L}[f^{(n)}](s) = s^n\mathcal{L}[f](s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s\,f^{(n-2)}(0) - f^{(n-1)}(0)\,. \qquad (6.8)$$

This means that if you know that a function $y(t)$ is n-times differentiable and that it and its first $n-1$ derivatives are of exponential order α as $t \to \infty$ then $Y(s) = \mathcal{L}[y](s)$ then

$$
\begin{aligned}
\mathcal{L}[y'](s) &= s\,Y(s) - y(0)\,, \\
\mathcal{L}[y''](s) &= s^2 Y(s) - s\,y(0) - y'(0)\,, \\
\mathcal{L}[y'''](s) &= s^3 Y(s) - s^2 y(0) - s\,y'(0) - y''(0)\,, \\
&\;\vdots \\
\mathcal{L}[y^{(n)}](s) &= s^n Y(s) - s^{n-1} y(0) - s^{n-2} y'(0) - \cdots - s\,y^{(n-2)}(0) - y^{(n-1)}(0)\,.
\end{aligned}
\tag{6.9}
$$

If $y(t)$ is the solution of an initial-value problem

$$
\begin{gathered}
\mathrm{D}^n y + a_1 \mathrm{D}^{n-1} y + \cdots + a_{n-1} \mathrm{D} y + a_n y = f(t)\,, \quad \text{where } \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,, \\
y(0) = y_0\,, \quad y'(0) = y_1\,, \quad \cdots, \quad y^{(n-1)}(0) = y_{n-1}\,,
\end{gathered}
$$

then you can use this fact to find $Y(s) = \mathcal{L}[y](s)$ in terms of $F(s) = \mathcal{L}[f](s)$ and the inital data y_0, y_1, $\cdot$, y_{n-1}. Indeed, the fact $\mathcal{L}$ is a linear operator implies that

$$
\mathcal{L}[\mathrm{D}^n y] + a_1 \mathcal{L}[\mathrm{D}^{n-1} y] + \cdots + a_{n-1} \mathcal{L}[\mathrm{D} y] + a_n \mathcal{L}[y] = \mathcal{L}[f]\,.
$$

If we use (6.9) then we find

$$
p(s) Y(s) = C(s) + F(s)\,,
$$

where $p(s)$ is the characteristic polynomial

$$
p(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n\,.
$$

and $C(s)$ is a polynomial that depends on the initial data

$$
\begin{aligned}
C(s) = {} & \big(s^{n-1} + a_1 s^{n-2} + \cdots + a_{n-2} s + a_{n-1}\big) y_0 \\
& + \big(s^{n-2} + a_1 s^{n-3} + \cdots + a_{n-3} s + a_{n-2}\big) y_1 \\
& \;\vdots \\
& + \big(s + a_1\big) y_{n-2} + y_{n-1}\,.
\end{aligned}
$$

Therefore one finds that

$$
Y(s) = \frac{C(s) + F(s)}{p(s)}\,.
$$

The next goal will be to determine $y(t)$ from $Y(s)$, but now let's see how to find $Y(s)$.

Example. Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$$y' - 2y = e^{5t}\,, \qquad y(0) = 3\,.$$

Because $\mathcal{L}[e^{5t}](s) = 1/(s-5)$, the Laplace transform of the initial-value problem gives

$$\mathcal{L}[y'](s) - 2\mathcal{L}[y](s) = \mathcal{L}[e^{5t}](s) = \frac{1}{s-5}\,,$$

where

$$\begin{aligned} \mathcal{L}[y](s) &= Y(s)\,, \\ \mathcal{L}[y'](s) &= s\,Y(s) - y(0) = s\,Y(s) - 3\,. \end{aligned}$$

Hence,

$$(s-2)Y(s) = \frac{1}{s-5} + 3 \qquad \Longrightarrow \qquad Y(s) = \frac{1}{(s-2)(s-5)} + \frac{3}{s-2}\,.$$

Example. Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$$y'' - 2y' - 8y = 0\,, \qquad y(0) = 3\,, \quad y'(0) = 7\,.$$

The Laplace transform of the initial-value problem gives

$$\mathcal{L}[y''](s) - 2\mathcal{L}[y'](s) - 8\mathcal{L}[y](s) = 0\,,$$

where

$$\begin{aligned} \mathcal{L}[y](s) &= Y(s)\,, \\ \mathcal{L}[y'](s) &= s\,Y(s) - y(0) = s\,Y(s) - 3\,, \\ \mathcal{L}[y''](s) &= s^2Y(s) - s\,y(0) - y'(0) = s^2Y(s) - 3s - 7\,. \end{aligned}$$

Hence,

$$(s^2 - 2s - 8)Y(s) = 3s + 1\,, \qquad \Longrightarrow \qquad Y(s) = \frac{3s+1}{(s^2 - 2s - 8)}\,.$$

Example. Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$$y'' + 4y = \sin(3t)\,, \qquad y(0) = y'(0) = 0\,.$$

Because $\mathcal{L}[\sin(3t)](s) = 3/(s^2+9)$, the Laplace transform of the initial-value problem gives

$$\mathcal{L}[y''](s) + 4\mathcal{L}[y](s) = \mathcal{L}[\sin(3t)](s) = \frac{3}{s^2 + 3^2} = \frac{3}{s^2 + 9}\,,$$

where

$$\begin{aligned} \mathcal{L}[y](s) &= Y(s)\,, \\ \mathcal{L}[y'](s) &= sY(s) - y(0) = sY(s)\,, \\ \mathcal{L}[y''](s) &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s)\,. \end{aligned}$$

Hence,

$$(s^2+4)Y(s) = \frac{3}{s^2+9}\,, \qquad \Longrightarrow \qquad Y(s) = \frac{3}{(s^2+4)(s^2+9)}\,.$$

Example. Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$$y'' + 4y = f(t)\,, \qquad y(0) = 7\,, \quad y'(0) = 5\,,$$

where

$$f(t) = \begin{cases} t^2 & \text{for } 0 \le t < 2\,, \\ 2t & \text{for } 2 \le t < 4\,, \\ 0 & \text{for } 4 \le t\,. \end{cases}$$

The Laplace transform of the initial-value problem gives

$$\mathcal{L}[y''](s) + 4\mathcal{L}[y](s) = F(s)\,,$$

where $F(s) = \mathcal{L}[f](s)$ and

$$\begin{aligned} \mathcal{L}[y](s) &= Y(s)\,, \\ \mathcal{L}[y'](s) &= sY(s) - y(0) = sY(s) - 7\,, \\ \mathcal{L}[y''](s) &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 7s - 5\,. \end{aligned}$$

To compute $F(s)$, first express $f(t)$ in terms of the unit step function as

$$\begin{aligned} f(t) &= \big(u(t) - u(t-2)\big)\,t^2 + \big(u(t-2) - u(t-4)\big)\,2t \\ &= u(t)t^2 - u(t-2)\,(t^2 - 2t) - u(t-4)2t \\ &= u(t)t^2 - u(t-2)\,\big((t-2)^2 + 2(t-2)\big) - u(t-4)\,\big(2(t-4) + 8\big)\,. \end{aligned}$$

In the last step above we express $f(t)$ as the sum of terms in the form $u(t-c_k)h_k(t-c_k)$ with $c_1 = 0$, $c_2 = 2$, and $c_3 = 4$, while $h_1(t) = t^2$, $h_2(t) = t^2 + 2t$, and $h_3(t) = 2t + 8$. This form allows you to use the fact $\mathcal{L}[u(t-c_k)h_k(t-c_k)](s) = e^{-c_k s}\mathcal{L}[h_k](s)$ to compute $F(s) = \mathcal{L}[f](s)$ as

$$\begin{aligned} F(s) &= \mathcal{L}[t^2](s) - \mathcal{L}\big[u(t-2)\,\big((t-2)^2 + 2(t-2)\big)\big](s) - \mathcal{L}\big[u(t-4)\,\big(2(t-4) + 8\big)\big](s) \\ &= \mathcal{L}[t^2](s) - e^{-2s}\mathcal{L}[t^2 + 2t](s) - e^{-4s}\mathcal{L}[2t + 8](s) \\ &= \frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3} + \frac{2}{s^2}\right) - e^{-4s}\left(\frac{2}{s^2} + \frac{8}{s}\right). \end{aligned}$$

Hence,

$$(s^2+4)Y(s) = 7s + 5 + \frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3} + \frac{2}{s^2}\right) - e^{-4s}\left(\frac{2}{s^2} + \frac{8}{s}\right),$$

whereby

$$Y(s) = \frac{7s+5}{s^2+9} + \frac{2}{s^3(s^2+4)} - e^{-2s}\left(\frac{2}{s^3(s^2+4)} + \frac{2}{s^2(s^2+4)}\right) - e^{-4s}\left(\frac{2}{s^2(s^2+4)} + \frac{8}{s\,(s^2+4)}\right).$$

The above example illustrates the four basic steps you must master in order to compute the Laplace transform of a function defined over $[0,\infty)$ by a list of cases in the form

$$f(t) = \begin{cases} f_1(t) & \text{for } 0 \le t < c_1\,, \\ f_2(t) & \text{for } c_1 \le t < c_2\,, \\ f_3(t) & \text{for } c_2 \le t < c_3\,, \\ \vdots & \qquad\vdots \\ f_{m-1}(t) & \text{for } c_{m-2} \le t < c_{m-1}\,, \\ f_m(t) & \text{for } c_{m-1} \le t < \infty\,, \end{cases}$$

where $0 < c_1 < c_2 < \cdots < c_{m-1}$.

The first step is to express $f(t)$ in terms of the unit step function as

$$\begin{aligned} f(t) = &\big(u(t) - u(t-c_1)\big)\, f_1(t) + \big(u(t-c_1) - u(t-c_2)\big)\, f_2(t) \\ &+ \big(u(t-c_2) - u(t-c_3)\big)\, f_3(t) + \cdots \\ &+ \big(u(t-c_{m-2}) - u(t-c_{m-1})\big)\, f_{m-1}(t) + u(t-c_{m-1})\, f_m(t)\,. \end{aligned}$$

This step becomes clear once you see that

$$u(t-c_{k-1}) - u(t-c_k) = \begin{cases} 1 & \text{for } c_{k-1} \le t < c_k\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

In other words, the function $u(t-c_{k-1}) - u(t-c_k)$ is a switch that turns on at $t = c_{k-1}$ and turns off at $t = c_k$.

The second step is to group terms that involve the same $u(t-c_k)$. Hence,

$$\begin{aligned} f(t) = &\, u(t)\, f_1(t) + u(t-c_1)\big(f_2(t) - f_1(t)\big) + u(t-c_2)\big(f_3(t) - f_2(t)\big) \\ &+ \cdots + u(t-c_{m-1})\big(f_m(t) - f_{m-1}(t)\big)\,. \end{aligned}$$

The next step is the hardest. You must express $f(t)$ in the form

$$\begin{aligned} f(t) = &\, f_1(t) + u(t-c_1)\, h_1(t-c_1) + u(t-c_2)\, h_2(t-c_2) \\ &+ \cdots + u(t-c_{m-1})\, h_{m-1}(t-c_{m-1})\,. \end{aligned}$$

In other words, you must espress each $f_{k+1}(t) - f_k(t)$ as a function of $t - c_k$. If you set

$$h_k(t) = f_{k+1}(c_k + t) - f_k(c_k + t)\,,$$

then it is clear that $f_{k+1}(t) - f_k(t) = h_k(t - c_k)$.

The final step is to use the fact that $\mathcal{L}[u(t - c_k)h_k(t - c_k)](s) = e^{-c_k s}\mathcal{L}[h_k](s)$ to compute $\mathcal{L}[f](s)$ as

$$\mathcal{L}[f](s) = \mathcal{L}[f_1](s) + e^{-c_1 s}\mathcal{L}[h_1](s) + e^{-c_2 s}\mathcal{L}[h_2](s) + \cdots + e^{-c_{m-1} s}\mathcal{L}[h_{m-1}](s)\,.$$

Often you will have to use identities to express each $h_k(t)$ in a form that allows you to compute $\mathcal{L}[h_k](s)$.

6.5: Inverse Transform. The process of determining $y(t)$ from $Y(s)$ is called taking the inverse Laplace transform. For us, this process will be one of expressing $Y(s)$ as a sum of terms that will allow us to read off $y(t)$ from a few basic forms. These basic forms are

$$\begin{aligned}
\mathcal{L}[t^n](s) &= \frac{n!}{s^{n+1}} && \text{for } s > 0\,,\\
\mathcal{L}[\cos(bt)](s) &= \frac{s}{s^2 + b^2} && \text{for } s > 0\,,\\
\mathcal{L}[\sin(bt)](s) &= \frac{b}{s^2 + b^2} && \text{for } s > 0\,,\\
\mathcal{L}[e^{at} f(t)](s) &= F(s - a) && \text{where } F(s) = \mathcal{L}[f(t)](s)\,,\\
\mathcal{L}[t^n f(t)](s) &= (-1)^n F^{(n)}(s) && \text{where } F(s) = \mathcal{L}[f(t)](s)\,,\\
\mathcal{L}[u(t - c)f(t - c)](s) &= e^{-cs} F(s) && \text{where } F(s) = \mathcal{L}[f(t)](s) \text{ and}\\
& && u(t) \text{ is the unit step function}\,.
\end{aligned}$$

One can use these to build up a much longer table of basic forms such as the one given in the book. However, the above table contains all the forms you really need. To illustrate this fact, we will compute the inverse Laplace transform of $Y(s)$ for the examples given in the previous section.

Example. Find the inverse Laplace transform $y(t)$ of

$$Y(s) = \frac{1}{(s - 2)(s - 5)} + \frac{3}{s - 2}\,.$$

By the partial fraction identity

$$\frac{1}{(s - 2)(s - 5)} = \frac{\frac{1}{3}}{s - 5} + \frac{-\frac{1}{3}}{s - 2}\,,$$

you can express $Y(s)$ as

$$Y(s) = \frac{\frac{1}{3}}{s-5} + \frac{\frac{8}{3}}{s-2}\,.$$

Hence,

$$y(t) = \tfrac{1}{3}e^{5t} + \tfrac{8}{3}e^{2t}\,.$$

Example. Find the inverse Laplace transform $y(t)$ of

$$Y(s) = \frac{3s+1}{(s^2-2s-8)}\,.$$

By the partial fraction identity

$$\frac{3s+1}{(s^2-2s-8)} = \frac{3s+1}{(s-4)(s+2)} = \frac{\frac{13}{6}}{s-4} + \frac{\frac{5}{6}}{s+2}\,,$$

you can express $Y(s)$ as

$$Y(s) = \frac{\frac{13}{6}}{s-4} + \frac{\frac{5}{6}}{s+2}\,.$$

Hence,

$$y(t) = \tfrac{13}{6}e^{4t} + \tfrac{5}{6}e^{-2t}\,.$$

Example. Find the inverse Laplace transform $y(t)$ of

$$Y(s) = \frac{3}{(s^2+4)(s^2+9)}\,.$$

By the partial fraction identity

$$\frac{3}{(z+4)(z+9)} = \frac{\frac{3}{5}}{(z+4)} + \frac{-\frac{3}{5}}{(z+9)}\,,$$

you can express $Y(s)$ as

$$Y(s) = \frac{\frac{3}{5}}{s^2+4} - \frac{\frac{3}{5}}{s^2+9}\,.$$

Hence,

$$y(t) = \tfrac{3}{10}\sin(2t) - \tfrac{1}{5}\sin(3t)\,.$$

Example. Find the inverse Laplace transform $y(t)$ of

$$\begin{aligned} Y(s) = \frac{7s+5}{s^2+4} + \frac{2}{s^3(s^2+4)} - e^{-2s}\left(\frac{2}{s^3(s^2+4)} + \frac{2}{s^2(s^2+4)}\right) \\ - e^{-4s}\left(\frac{2}{s^2(s^2+4)} + \frac{8}{s\,(s^2+4)}\right). \end{aligned}$$

You first derive the partial fraction identities

$$\begin{aligned}
\frac{7s+5}{s^2+4} &= \frac{7s}{s^2+4} + \frac{5}{s^2+4}\,,\\
\frac{2}{s^3(s^2+4)} &= \frac{\frac{1}{2}}{s^3} - \frac{\frac{1}{8}}{s} + \frac{\frac{1}{8}s}{s^2+4}\,,\\
\frac{2}{s^2(s^2+4)} &= \frac{\frac{1}{2}}{s^2} - \frac{\frac{1}{2}}{s^2+4}\,,\\
\frac{8}{s\,(s^2+4)} &= \frac{2}{s} - \frac{2s}{s^2+4}\,.
\end{aligned}$$

The first of these is straightforward. The second can be approached by observing that its left-hand side is an odd function of s, so that the partial fraction decomposition must consist of odd functions of s. In particular, the partial fraction decomposition must have the form

$$\frac{2}{s^3(s^2+4)} = \frac{A}{s^3} + \frac{B}{s} + \frac{Cs}{s^2+4}\,.$$

The third can be approached by observing that its left-hand side is an even function of s, so that the partial fraction decomposition must consist of even functions of s. In particular, the partial fraction decomposition must have the form

$$\frac{2}{s^2(s^2+4)} = \frac{A}{s^2} + \frac{B}{s^2+4}\,.$$

The last partial fraction identity is just $4s$ times the third. These partial fraction identites allow you to express $Y(s)$ as

$$\begin{aligned}
Y(s) = \frac{7s}{s^2+4} + \frac{5}{s^2+4} &+ (1-e^{-2s})\left(\frac{\frac{1}{2}}{s^3} - \frac{\frac{1}{8}}{s} + \frac{\frac{1}{8}s}{s^2+4}\right)\\
&+ (e^{-2s}-e^{-4s})\left(\frac{\frac{1}{2}}{s^2} - \frac{\frac{1}{2}}{s^2+4}\right) + e^{-4s}\left(\frac{2}{s} - \frac{2s}{s^2+4}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
y(t) = 7\cos(2t) &+ \tfrac{5}{2}\cos(2t) + \big(\tfrac{1}{4}t^2 - \tfrac{1}{8} + \tfrac{1}{8}\cos(2t)\big)\\
&- u(t-2)\big(\tfrac{1}{4}(t-2)^2 - \tfrac{1}{8} + \tfrac{1}{8}\cos(2(t-2))\big) + u(t-2)\big(\tfrac{1}{2}(t-2) - \tfrac{1}{4}\sin(2(t-2))\big)\\
&+ u(t-4)\big(\tfrac{1}{2}(t-4) - \tfrac{1}{4}\sin(2(t-4))\big) + u(t-4)\big(2 - 2\cos(2(t-4))\big)\,.
\end{aligned}$$