Quiz 1 Solutions, Math 246, Professor David Levermore
Tuesday, 9 September 2008

(1) [4] Determine the order of the given differential equation; also state whether the equation is linear or nonlinear:

(a) $t^2 \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + t\,\frac{\mathrm{d}y}{\mathrm{d}t} - 4y = t^3$ **Solution:** second-order, linear

(b) $\frac{\mathrm{d}^3 v}{\mathrm{d}t^3} + v\frac{\mathrm{d}v}{\mathrm{d}t} + 4v = \cos(t)$ **Solution:** third-order, nonlinear

(2) [6] Solve the given initial-value problems:

(a) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^x}{2y}\,, \qquad y(0) = -2$

Solution: This problem is separable. Its separated form is

$$2y\,\mathrm{d}y = e^x\,\mathrm{d}x\,.$$

Integrating both sides yields

$$y^2 = e^x + c\,.$$

Applying the initial condition gives

$$(-2)^2 = e^0 + c\,,$$

which implies $c = 4 - 1 = 3$. The solution is therefore

$$y = -\sqrt{e^x + 3}\,, \qquad \text{for every } x\,,$$

where the negative root is taken to satisfy the initial condition.

(b) $t\,\frac{\mathrm{d}z}{\mathrm{d}t} = 5t^2 - 3z\,, \qquad z(1) = 5$

Solution: This problem is linear. Its normal form is

$$\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{3}{t}\,z = 5t\,.$$

An integrating factor is $e^{A(t)}$ where $A'(t) = 3/t$. Setting $A(t) = 3\log(t)$, we find that $e^{A(t)} = t^3$. Hence, the problem has the intgrating factor form

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(t^3 z\big) = t^3 \cdot 5t = 5t^4\,.$$

Integrating both sides yields

$$t^3 z = t^5 + c\,.$$

Applying the initial condition gives

$$1^3 \cdot 5 = 1^5 + c\,,$$

which implies $c = 5 - 1 = 4$. The solution is therefore

$$z = t^2 + \frac{4}{t^3}\,, \qquad \text{for every } t > 0\,.$$