Quiz 4 Solutions, Math 246, Professor David Levermore
Tuesday, 7 October 2008

(1) [4] What is the interval of existence for the solution to each of the following initial-value problems.

(a) $(3t - t^2)\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} + \cos(t)\dfrac{\mathrm{d}x}{\mathrm{d}t} + e^t x = e^{2t}\,, \quad x(4) = 0\,, \quad x'(4) = 7\,.$

Solution: The linear normal form is $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} + \dfrac{\cos(t)}{t(3-t)}\dfrac{\mathrm{d}x}{\mathrm{d}t} + \dfrac{e^t}{t(3-t)}x = \dfrac{e^{2t}}{t(3-t)}$. The coefficients and forcing are defined and continuous everywhere except $t = 0$ and $t = 3$. The initial time is $t = 4$. The interval of existence is therefore $(3, \infty)$.

(b) $\dfrac{\mathrm{d}^4z}{\mathrm{d}t^4} + z = \cot(t)\,, \quad z(5) = z'(5) = z''(5) = z'''(5) = 0\,.$

Solution: The equation is already in linear normal form. The coefficients and forcing are defined and continuous everywhere except $t = k\pi$ for some integer k. The initial time is $t = 5$. The interval of existence is therefore $(\pi, 2\pi)$.

(2) [3] Compute the Wronskian $W[Y_1, Y_2](t)$ of the functions $Y_1(t) = e^{-2t}$ and $Y_2(t) = e^{3t}$. (Evaluate the determinant and simplify.)

Solution: Because $Y_1'(t) = -2e^{-2t}$ and $Y_2'(t) = 3e^{3t}$, the Wronskian is

$$W[Y_1, Y_2](t) = \det\begin{pmatrix} Y_1(t) & Y_2(t) \\ Y_1'(t) & Y_2'(t) \end{pmatrix} = \det\begin{pmatrix} e^{-2t} & e^{3t} \\ -2e^{-2t} & 3e^{3t} \end{pmatrix}$$
$$= 3e^{-2t}e^{3t} - (-2)e^{-2t}e^{3t} = 5e^t\,.$$

(3) [3] The functions e^{-2t} and e^{2t} are linearly independent solutions of $\dfrac{\mathrm{d}^2y}{\mathrm{d}t^2} - 4y = 0$. Find the solution $Y(t)$ that satisfies the initial conditions $Y(0) = 8$, $Y'(0) = 4$.

Solution: Let $Y(t) = c_1e^{-2t} + c_2e^{2t}$. Then $Y'(t) = -2c_1e^{-2t} + 2c_2e^{2t}$. To satisfy the initial conditions one needs

$$8 = Y(0) = c_1 + c_2\,, \qquad 4 = Y'(0) = -2c_1 + 2c_2\,.$$

Divide the second equation by 2 and add it to the first to obtain $10 = 2c_2$, whereby $c_2 = 5$. It then follows from the first equation that $c_1 = 8 - c_2 = 8 - 5 = 3$. The solution of the initial value problem is therefore

$$Y(t) = 3e^{-2t} + 5e^{2t}\,.$$