Quiz 5 Solutions, Math 246, Professor David Levermore
Tuesday, 14 October 2008

(1) [6] Give a general solution to each of the following.

(a) $\dfrac{\mathrm{d}^3x}{\mathrm{d}t^3} + 6\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} + 13\dfrac{\mathrm{d}x}{\mathrm{d}t} = 0$

Solution: The coefficients are constant. The characteristic polynomial is

$$\begin{aligned} p(z) &= z^3 + 6z^2 + 13z = z(z^2 + 6z + 13) = z\big((z+3)^2 + 4\big) \\ &= z\big((z+3)^2 + 2^2\big)\,. \end{aligned}$$

Its three roots are $z = 0, -3 + i2, -3 - i2$. A general solution is therefore

$$x = c_1 + c_2 e^{-3t}\cos(2t) + c_3 e^{-3t}\sin(2t)\,.$$

(b) $\dfrac{\mathrm{d}^4y}{\mathrm{d}t^4} - 16y = 0$

Solution: The coefficients are constant. The characteristic polynomial is

$$\begin{aligned} p(z) &= z^4 - 16 = (z^2 - 4)(z^2 + 4) = (z-2)(z+2)(z^2+4) \\ &= (z-2)(z+2)(z^2 + 2^2)\,. \end{aligned}$$

Its four roots are $z = 2, -2, i2, -i2$. A general solution is therefore

$$y = c_1 e^{2t} + c_2 e^{-2t} + c_3\cos(2t) + c_4\sin(2t)\,.$$

(2) [4] An eighth order linear differential operator L with constant coefficients has a characteristic polynomial $p(z)$ with roots $z = 2 + i5, 2 + i5, 2 - i5, 2 - i5, 3, 3, 3, 3$. Give a general solution to the equation $\mathrm{L}y = 0$.

Solution: A general solution is

$$\begin{aligned} y &= c_1 e^{2t}\cos(5t) + c_2 e^{2t}\sin(5t) + c_3 t\, e^{2t}\cos(5t) + c_4 t\, e^{2t}\sin(5t) \\ &\quad + c_5 e^{3t} + c_6 t\, e^{3t} + c_7 t^2 e^{3t} + c_8 t^3 e^{3t}\,. \end{aligned}$$

The complex conjugate pair of roots $2 \pm i5$ with multiplicity 2 contributes

$$e^{2t}\cos(5t)\,, \qquad e^{2t}\sin(5t)\,, \qquad t\, e^{2t}\cos(5t)\,, \qquad t\, e^{2t}\sin(5t)\,.$$

The real root 3 with multiplicity 4 contributes

$$e^{3t}\,, \qquad t\, e^{3t}\,, \qquad t^2 e^{3t}\,, \qquad t^3 e^{3t}\,.$$