Quiz 6 Solutions, Math 246, Professor David Levermore
Tuesday, 21 October 2008

(1) [5] Find a general solution of $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} + \dfrac{\mathrm{d}x}{\mathrm{d}t} - 2x = e^t - 4t$.

Solution. The characteristic polynomial is $P(z) = z^2 + z - 2 = (z-1)(z+2)$ and has roots 1 and -2. The solution of the associated homogeneous problem is

$$X_H(t) = c_1 e^t + c_2 e^{-2t}\,.$$

The forcing term e^t has degree 0 and characteristic 1 which is a root of $P(z)$ of multiplicity 1. The forcing term $-4t$ has degree 1 and characteristic 0 which is a root of $P(z)$ of multiplicity 0. We therefore need the KEY identity and its first derivative with respect to z:

$$\begin{aligned} \mathrm{L}\big(e^{zt}\big) &= (z^2 + z - 2)e^{zt}\,, \\ \mathrm{L}\big(t\,e^{zt}\big) &= (z^2 + z - 2)t\,e^{zt} + (2z+1)e^{zt}\,. \end{aligned}$$

Evaluating these at $z = 1$ gives $\mathrm{L}(e^t) = 0$ and $\mathrm{L}(t\,e^t) = 3e^t$, whereby $\mathrm{L}(\frac{1}{3}t\,e^t) = e^t$. Evaluating these at $z = 0$ gives $\mathrm{L}(1) = -2$ and $\mathrm{L}(t) = -2t + 1$, which implies $\mathrm{L}(t + \frac{1}{2}) = -2t$. Hence, $X_P(t) = \frac{1}{3}t\,e^t + 2t + 1$. A general solution is

$$x = c_1 e^t + c_2 e^{-2t} + \tfrac{1}{3}t\,e^t + 2t + 1\,.$$

Alternatively, for the forcing term e^t you set $X_P(t) = At\,e^t$, plug it into the equation, and solve for A; for the forcing term $-4t$ you set $X_P(t) = A_0 t + A_1$, plug it into the equation, and solve for A_0 and A_1. You obtain the same general solution.

(2) [5] Find a general solution of $\dfrac{\mathrm{d}^2y}{\mathrm{d}t^2} + 4\dfrac{\mathrm{d}y}{\mathrm{d}t} + 8y = 4\cos(2t)$.

Solution. The characteristic polynomial is $P(z) = z^2 + 4z + 8 = (z+2)^2 + 2^2$ and has roots $-2 \pm i2$. The solution of the associated homogeneous problem is

$$Y_H(t) = c_1 e^{-2t}\cos(2t) + c_2 e^{-2t}\sin(2t)\,.$$

The forcing term $4\cos(2t)$ has degree 0 and characteristic $i2$ which is a root of $P(z)$ of multiplicity 0. We therefore only need the KEY identity:

$$\mathrm{L}\big(e^{zt}\big) = (z^2 + 4z + 8)e^{zt}\,.$$

Evaluating this at $z = i2$ gives $\mathrm{L}(e^{i2t}) = (4 + i8)e^{i2t}$, whereby

$$\mathrm{L}\left(\frac{e^{i2t}}{1+i2}\right) = 4e^{i2t}\,.$$

Because $4\cos(2t) = \mathrm{Re}\big(4e^{i2t}\big)$, you see that

$$Y_P(t) = \mathrm{Re}\left(\frac{e^{i2t}}{1+i2}\right) = \tfrac{1}{5}\,\mathrm{Re}\big((1-i2)e^{i2t}\big) = \tfrac{1}{5}\cos(2t) + \tfrac{2}{5}\sin(2t)\,.$$

A general solution is

$$y = c_1 e^{-2t}\cos(2t) + c_2 e^{-2t}\sin(2t) + \tfrac{1}{5}\cos(2t) + \tfrac{2}{5}\sin(2t)\,.$$

Alternatively, you can set $Y_P(t) = A\cos(2t) + B\sin(2t)$, plug it into the equation, and solve for A and B. You obtain the same general solution.