Quiz 7 Solutions, Math 246, Professor David Levermore
Tuesday, 28 October 2008

(1) [4] Given the fact that the associated Green function is $g(t) = \sin(t)$, use the Green function method to find a particular solution of

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + x = \frac{1}{\cos(t)} \,.$$

Solution: A particular solution $X_P(t)$ is given by

$$\begin{aligned} X_P(t) &= \int_0^t \sin(t-s)\,\frac{1}{\cos(s)}\,\mathrm{d}s \\ &= \int_0^t \big(\sin(t)\cos(s) - \cos(t)\sin(s)\big)\,\frac{1}{\cos(s)}\,\mathrm{d}s \\ &= \sin(t)\int_0^t \frac{\cos(s)}{\cos(s)}\,\mathrm{d}s - \cos(t)\int_0^t \frac{\sin(s)}{\cos(s)}\,\mathrm{d}s \\ &= \sin(t)\,t + \cos(t)\,\log(|\cos(t)|) \,. \end{aligned}$$

(2) [6] The functions $t-1$ and e^{-t} are solutions of the equation

$$t\,\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + (t-1)\frac{\mathrm{d}y}{\mathrm{d}t} - y = 0 \,.$$

(You do not have to check this fact.) Find a general solution of

$$t\,\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + (t-1)\frac{\mathrm{d}y}{\mathrm{d}t} - y = -t^2e^{-t} \,.$$

Solution: Because this equation has variable coefficients, you must use the variation of parameters method. Divide the equation by t to bring it into its normal form

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{t-1}{t}\,\frac{\mathrm{d}y}{\mathrm{d}t} - \frac{1}{t}\,y = -t\,e^{-t} \,.$$

Seek a solution in the form

$$y = u_1(t)(t-1) + u_2(t)e^{-t} \,,$$

where

$$\begin{aligned} u_1'(t)(t-1) + u_2'(t)e^{-t} &= 0 \,, \\ u_1'(t)1 - u_2'(t)e^{-t} &= -t\,e^{-t} \,. \end{aligned}$$

The solution of this system is

$$u_1'(t) = -e^{-t} \,, \qquad u_2'(t) = t - 1 \,,$$

which can be integrated to obtain

$$u_1(t) = c_1 + e^{-t} \,, \qquad u_2(t) = c_2 + \tfrac{1}{2}(t-1)^2 \,.$$

A general solution is therefore

$$y = c_1(t-1) + c_2e^{-t} + (t-1)e^{-t} + \tfrac{1}{2}(t-1)^2e^{-t} \,.$$