Quiz 8 Solutions, Math 246, Professor David Levermore
Tuesday, 11 November 2008

Short Table: $\mathcal{L}[\sin(bt)](s) = \dfrac{b}{s^2+b^2}$ for $s > 0$, $\mathcal{L}[e^{at}](s) = \dfrac{1}{s-a}$ for $s > a$.

(1) [3] Use the definition of the Laplace transform to compute $\mathcal{L}[f](s)$ for $f(t) = u(t-5)$, where u is the unit step function.

Solution: By the definitions of the Laplace transform and the unit step function

$$\mathcal{L}[f](s) = \lim_{T\to\infty} \int_0^T e^{-st} f(t)\,\mathrm{d}t = \lim_{T\to\infty} \int_5^T e^{-st}\,\mathrm{d}t\,.$$

The above limit diverges for $s \leq 0$. For $s > 0$

$$\int_5^T e^{-st}\,\mathrm{d}t = -\frac{e^{-st}}{s}\bigg|_5^T = \frac{e^{-s5}}{s} - \frac{e^{-sT}}{s}\,,$$

whereby

$$\mathcal{L}[f](s) = \lim_{T\to\infty}\left[\frac{e^{-s5}}{s} - \frac{e^{-sT}}{s}\right] = \frac{e^{-5s}}{s} \quad \text{for } s > 0\,.$$

(2) [4] Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$y'' + 9y = \sin(2t)\,, \qquad y(0) = y'(0) = 0\,.$ DO NOT solve for $y(t)$, just $Y(s)$!

Solution: The Laplace transform of the initial-value problem and item 1 in the table at the top of the page with $b = 2$ gives

$$\mathcal{L}[y''](s) + 9\mathcal{L}[y](s) = \mathcal{L}[\sin(2t)](s) = \frac{2}{s^2+2^2} = \frac{2}{s^2+4}\,,$$

where

$$\begin{aligned}\mathcal{L}[y](s) &= Y(s)\,,\\ \mathcal{L}[y''](s) &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s)\,.\end{aligned}$$

Hence,

$$(s^2+9)Y(s) = \frac{2}{s^2+4}\,, \qquad \Longrightarrow \qquad Y(s) = \frac{2}{(s^2+9)(s^2+4)}\,.$$

(3) [3] Find the inverse Laplace transform $y(t)$ of the function $Y(s) = \dfrac{6s}{s^2+2s-8}$.

Solution: Partial fractions and item 2 in the table at the top of the page with $a = 2$ and with $a = -4$ gives

$$\begin{aligned}Y(s) &= \frac{6s}{s^2+2s-8} = \frac{6s}{(s-2)(s+4)} = \frac{2}{s-2} + \frac{4}{s+4}\\ &= 2\mathcal{L}[e^{2t}](s) + 4\mathcal{L}[e^{-4t}](s) = \mathcal{L}\big[2e^{2t} + 4e^{-4t}\big](s)\,.\end{aligned}$$

Therefore

$$y(t) = \mathcal{L}^{-1}[Y](t) = 2e^{2t} + 4e^{-4t}\,.$$