Quiz 10 Solutions, Math 246, Professor David Levermore
Tuesday, 25 November 2008

(1) [2] Given that $e^{t\mathbf{A}} = \begin{pmatrix} \cos(3t) & -\sin(3t) \\ \sin(3t) & \cos(3t) \end{pmatrix}$,

solve the initial-value problem $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x}\,, \quad \mathbf{x}(0) = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

Solution.

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}(0) = \begin{pmatrix} \cos(3t) & -\sin(3t) \\ \sin(3t) & \cos(3t) \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 5\cos(3t) - 2\sin(3t) \\ 5\sin(3t) + 2\cos(3t) \end{pmatrix}.$$

(2) [4] Let $\mathbf{A} = \begin{pmatrix} 8 & -5 \\ 4 & 0 \end{pmatrix}$. Compute $e^{t\mathbf{A}}$.

Solution. The characteristic polynomial is $p(z) = z^2 - 8z + 20 = (z-4)^2 + 2^2$. Hence,

$$\begin{aligned} e^{t\mathbf{A}} &= \mathbf{I}\, e^{4t}\cos(2t) + (\mathbf{A} - 4\mathbf{I})\, e^{4t}\frac{\sin(2t)}{2} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e^{4t}\cos(2t) + \begin{pmatrix} 4 & -5 \\ 4 & -4 \end{pmatrix} e^{4t}\frac{\sin(2t)}{2} \\ &= e^{4t}\begin{pmatrix} \cos(2t) + 2\sin(2t) & -\frac{5}{2}\sin(2t) \\ 2\sin(2t) & \cos(2t) - 2\sin(2t) \end{pmatrix}. \end{aligned}$$

(3) [4] Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} t^2 + 1 \\ t \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t \\ 1 \end{pmatrix}$.

(a) Compute the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2](t)$.

Solution.

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \det\begin{pmatrix} t^2 + 1 & t \\ t & 1 \end{pmatrix} = (t^2 + 1) - t^2 = 1\,.$$

(b) Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x}$.

Solution. Let $\mathbf{\Psi}(t) = \begin{pmatrix} t^2 + 1 & t \\ t & 1 \end{pmatrix}$. Because $\dfrac{\mathrm{d}\mathbf{\Psi}}{\mathrm{d}t}(t) = \mathbf{A}(t)\mathbf{\Psi}(t)$, one has

$$\begin{aligned} \mathbf{A}(t) &= \frac{\mathrm{d}\mathbf{\Psi}}{\mathrm{d}t}(t)\,\mathbf{\Psi}(t)^{-1} = \begin{pmatrix} 2t & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t^2 + 1 & t \\ t & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 2t & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -t \\ -t & t^2 + 1 \end{pmatrix} = \begin{pmatrix} t & 1 - t^2 \\ 1 & -t \end{pmatrix}. \end{aligned}$$