Quiz 11 Solutions, Math 246, Professor David Levermore
Tuesday, 2 December 2008

(1) [4] $\mathbf{A} = \begin{pmatrix} 4 & 3 \\ 4 & 0 \end{pmatrix}$ has eigenvalues -2 and 6. Find an eigenvector for each eigenvalue.

Solution: One has $\mathbf{A} + 2\mathbf{I} = \begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix}$ and $\mathbf{A} - 6\mathbf{I} = \begin{pmatrix} -2 & 3 \\ 4 & -6 \end{pmatrix}$.
The eigenvectors $\mathbf{v}_1$ associated with -2 satisfy $(\mathbf{A} + 2\mathbf{I})\mathbf{v}_1 = 0$. You can either solve this system or simply read-off from a nonzero column of $\mathbf{A} - 6\mathbf{I}$ that these have the form

$$\mathbf{v}_1 = \alpha_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \text{for some } \alpha_1 \neq 0\,.$$

The eigenvectors $\mathbf{v}_2$ associated with 6 satisfy $(\mathbf{A} - 6\mathbf{I})\mathbf{v}_2 = 0$. You can either solve this system or simply read-off from a nonzero column of $\mathbf{A} + 2\mathbf{I}$ that these have the form

$$\mathbf{v}_2 = \alpha_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \text{for some } \alpha_2 \neq 0\,.$$

(2) [3] The 2×2 matrix $\mathbf{A}$ has the real eigenpairs $\left(-1\,, \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right)$ and $\left(2\,, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$.
Sketch a phase portrait for the system $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x}$. Indicate typical trajectories.
Solution: Because the eigenvalues of $\mathbf{A}$ are real and of opposite sign, the phase portrait is a *saddle*.
Trajectories on the line $c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ move towards the origin.
Trajectories on the line $c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ move away from the origin.

(3) [3] Sketch a phase portrait for the system $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x}\,,$ where $\mathbf{A} = \begin{pmatrix} 0 & -2 \\ 5 & 2 \end{pmatrix} \mathbf{x}\,.$

Solution: The characteristic polynomial of $\mathbf{A}$ is

$$p(z) = z^2 - \operatorname{to}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z + 10 = (z-1)^2 + 3^3\,.$$

The eigenvalues of $\mathbf{A}$ are therefore $1 + i3$ and $1 - i3$, whereby the phase portrait is a *spiral source.* Because $a_{21} = 5 > 0$ the spiral will be *counterclockwise.*

Alternatively, you can read off the direction of the spiral by noticing that

$$\text{for} \quad \mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{one has} \quad \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x} = \begin{pmatrix} 0 & -2 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix},$$

so the spiral will be counterclockwise. One can use any point $\mathbf{x}$ to reach the same conclusion.