**AMSC 674 Final Exam, Spring 2009**
**Professor David Levermore**
**Due 5pm Wednesday May 13**

(1) Let $\Omega \subset \mathbb{R}^D$ be a smooth bounded domain. Consider the boundary-value problem

$$\begin{aligned} -\nabla_x \cdot \big(A(x)\nabla_x u\big) + c(x)u &= f(x) \qquad \text{in } \Omega\,, \\ n(x)\cdot\big(A(x)\nabla_x u\big) + b(x)u &= g(x) \qquad \text{on } \partial\Omega\,, \end{aligned}$$

where $A$, $c$, and $f$ are smooth over $\overline{\Omega}$, $b$ and $g$ are smooth over $\partial\Omega$, $n$ is the outward unit normal on $\partial\Omega$, $b$ and $c$ are nonnegative, and the matrix-valued function $A$ is symmetric and satifies the uniformly ellipticity condition. Give a weak formulation of this problem and use the Lax-Milgram theorem to show the existence of a weak solution in $H^1(\Omega)$ when either $b$ or $c$ is nontrivial.

(2) Let $p \in (0,\infty)$. Consider $u(x) = |x|^{-\frac{D}{p}}$ over $\mathbb{R}^D$. Show that $u \in L^p_w(\mathrm{d}m)$ where $\mathrm{d}m$ is the usual Lebesgue measure, and that it is in no other weak Lebesgue space. Compute $[u]_{L^p_w}$. Compute $\|u\|_{L^p_w}$ for $p \in (1,\infty)$.

(3) Let $u$ be a smooth solution of the initial-value problem over $\mathbb{R}^D \times [0,\infty)$ given by

$$\partial_t u = \Delta_x u - \sin(u)\,, \qquad u\big|_{t=0} = u_I\,.$$

Prove that if $u_I$ is nonegative then so is $u$.

(4) Let $\Omega \subset \mathbb{R}^D$ be a smooth bounded domain. Let $p \in [1,\infty)$. Prove that there does not exist a bounded operator $T : L^p(\Omega) \to L^p(\partial\Omega)$ such that $Tu = u\big|_{\partial\Omega}$ whenever $u \in C(\overline{\Omega}) \cap L^p(\Omega)$.

(5) Let $X$ be a Banach space and $S(t)$ be a strongly continuous semigroup on $X$ with generator $A$. Let $\mathrm{Dom}(A) \subset X$ be the domain of $A$. For every $k \in \mathbb{Z}_+$ inductively define

$$\mathrm{Dom}(A^{k+1}) = \big\{ u \in \mathrm{Dom}(A^k) \,:\, Au \in \mathrm{Dom}(A^k) \big\}\,.$$

Show that if $u \in \mathrm{Dom}(A^k)$ for some $k \in \mathbb{Z}_+$ then $S(t)u \in \mathrm{Dom}(A^k)$ for every $t > 0$.

(6) Consider the initial-value problem over $\mathbb{R}^D \times [0,\infty)$ formally given by

$$\partial_{tt} u + \Delta_x^2 u = 0\,, \qquad u\big|_{t=0} = u_I\,, \quad \partial_t u\big|_{t=0} = v_I\,.$$

Formulate and prove a well-posedness result when $u_I$ and $v_I$ lie in any suitable Sololev spaces $H^r(\mathbb{R}^D)$ and $H^s(\mathbb{R}^D)$ respectively. (You may choose $r$ and $s$ or relate them.)

(7) Let $A = \sqrt{-\Delta_x}$. For every $\tau > 0$, $s > 0$, and $r \geq 0$ define

$$\mathrm{Dom}\big(e^{\tau A^{\frac{1}{s}}}, H^r(\mathbb{T}^D)\big) = \Big\{ w \in H^r(\mathbb{T}^D) \,:\, e^{\tau A^{\frac{1}{s}}} w \in H^r(\mathbb{T}^D) \Big\}\,.$$

Show that $\mathrm{Dom}\big(e^{\tau A^{\frac{1}{s}}}, H^r(\mathbb{T}^D)\big)$ is an algebra (a linear space that is closed under multiplication) when $s \geq 1$ and $r > \frac{D}{2}$.