INTEGRAL OPERATORS

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Abstract. We give bounds on integral operators that act either on classical Lebesgue spaces or on weak Lebesgue spaces. These include Hölder-Young bounds for operators with regular kernels, Hardy-Littlewood bounds for operators with weakly singular kernels, and Calderon-Zygmund bounds for strongly singular convolution operators over Euclidean space.

1. Introduction

Let $(X, \Sigma_\mu, \mathrm{d}\mu)$ and $(Y, \Sigma_\nu, \mathrm{d}\nu)$ be positive σ-finite measure spaces. Let $M(\mathrm{d}\mu)$ and $M(\mathrm{d}\nu)$ be the spaces of all complex-valued $\mathrm{d}\mu$-measurable and $\mathrm{d}\nu$-measurable functions respectively. As usual, functions in these spaces are considered identical if they are equal almost everywhere. We consider linear integral operators $\mathcal{K}$ of the form

$$\mathcal{K}u(y) = \int k(x,y)\, u(x)\, \mathrm{d}\mu(x)\,, \tag{1.1}$$

where the kernel k is a complex-valued measurable function with respect to the σ-algebra $\Sigma_{\mu\otimes\nu}$. We seek conditions on k that imply the operator $\mathcal{K}$ is bounded or even compact from $\mathcal{X}$ to $\mathcal{Y}$ where $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ are Banach spaces of functions that are contained within $M(\mathrm{d}\mu)$ and $M(\mathrm{d}\nu)$ respectively. We will first obtain such results for classical Lebesgue spaces — namely, for cases where

$$\mathcal{X} = L^p(\mathrm{d}\mu) \text{ and } \mathcal{Y} = L^{q^*}(\mathrm{d}\nu) \text{ for some } p, q^* \in [1,\infty]\,.$$

We will then extend these results to weak Lebesgue spaces — namely, to cases where

$$\mathcal{X} = L^p_w(\mathrm{d}\mu) \text{ or } \mathcal{Y} = L^{q^*}_w(\mathrm{d}\nu) \text{ for some } p, q^* \in (1,\infty)\,.$$

These notes will assume that you have some familiarity with the classical Lebesgue spaces, but will be self-contained regarding the weak Lebesgue spaces.

1.1. **Bounded Linear Operators.** Recall that a linear operator $\mathcal{K}$ that maps a normed space $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ into a normed space $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ is said to be *bounded* if it maps bounded subsets of $\mathcal{X}$ into bounded bounded subsets of $\mathcal{Y}$. It is easy to show that this is equivalent to the property that there exists a constant $C < \infty$ such that

$$\|\mathcal{K}u\|_{\mathcal{Y}} \leq C\, \|u\|_{\mathcal{X}} \quad \text{for every } u \in \mathcal{X}\,. \tag{1.2}$$

It easy to see that every bounded linear operator is continuous. It is not hard to show that the converse is also true. The notions of bounded and continuous thereby coincide for linear operators acting between normed spaces. It is customary to prefer the terminology *bounded linear operator* over that of *continuous linear operator*. The reason for this preference is the fact that the hard part of showing a linear operator is continuous is usually establishing the bound (1.2).

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A Banach space is a complete normed space. When $\mathcal{Y}$ is a Banach space and $\mathcal{K}$ is defined over a dense linear subspace of $\mathcal{X}$, it suffices to establish (1.2) for every u in that subspace. Because (1.2) implies that $\mathcal{K}$ is uniformly continuous over bounded subsets of $\mathcal{X}$, there is a unique extension of $\mathcal{K}$ to every $u \in \mathcal{X}$ so that (1.2) holds.

The space of all bounded linear operators from a normed space $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ into a normed space $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ is denoted $B(\mathcal{X},\mathcal{Y})$. For each $\mathcal{K} \in B(\mathcal{X},\mathcal{Y})$ we define $\|\mathcal{K}\|_{B(\mathcal{X},\mathcal{Y})}$ to be the infimum of all constants C such that (1.2) holds. It is easy to check that $B(\mathcal{X},\mathcal{Y})$ is a linear space and that $\|\cdot\|_{B(\mathcal{X},\mathcal{Y})}$ is a norm on $B(\mathcal{X},\mathcal{Y})$. Moreover, if $\mathcal{S}$ is any dense linear subspace of $\mathcal{X}$ then one can show that

$$\|\mathcal{K}\|_{B(\mathcal{X},\mathcal{Y})} = \sup_{u\in\mathcal{S}} \big\{ \|\mathcal{K}u\|_{\mathcal{Y}} \,:\, \|u\|_{\mathcal{X}} = 1 \big\} = \sup_{u\in\mathcal{S}} \left\{ \frac{\|\mathcal{K}u\|_{\mathcal{Y}}}{\|u\|_{\mathcal{X}}} \,:\, u \neq 0 \right\}. \tag{1.3}$$

Finally, $B(\mathcal{X},\mathcal{Y})$ equipped with this norm is a Banach space whenever $\mathcal{Y}$ is a Banach space. In particular, the dual space of $\mathcal{X}$, defined by $\mathcal{X}^* = B(\mathcal{X},\mathbb{C})$, is always a Banach space.

1.2. **Compact Linear Operators.** Recall that a linear operator $\mathcal{K}$ that maps a normed space $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ into a normed space $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ is said to be *compact* if it maps bounded subsets of $\mathcal{X}$ into totally bounded subsets of $\mathcal{Y}$. Because every totally bounded subset of a metric space (hence, of a normed space) is also bounded, it is therefore clear that every compact linear operator is also a bounded linear operator. It is easy to show that $\mathcal{K}$ being compact is equivalent to the property that $\mathcal{K}$ maps the unit ball of $\mathcal{X}$ into a totally bounded subset of $\mathcal{Y}$.

The following theorem is at the heart of many arguments that a given linear operator is compact. It states that any bounded linear operator which can be approximated by compact linear operators is also compact.

Theorem 1.1. *Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be normed spaces. Let $\mathcal{K} \in B(\mathcal{X},\mathcal{Y})$. If there exists a sequence $\{\mathcal{K}_n\}_{n\in\mathbb{N}} \subset B(\mathcal{X},\mathcal{Y})$ such that*

(1) *$\mathcal{K}_n \to \mathcal{K}$ in $B(\mathcal{X},\mathcal{Y})$ as $n \to \infty$,*
(2) *each $\mathcal{K}_n$ is compact,*

then $\mathcal{K}$ is compact.

Proof. Let $B_{\mathcal{X}}$ be the unit ball in $\mathcal{X}$. We will show that the set $\mathcal{K}B_{\mathcal{X}} \subset \mathcal{Y}$ is totally bounded. Let $\epsilon > 0$. Because $\mathcal{K}_n \to \mathcal{K}$ in $B(\mathcal{X},\mathcal{Y})$ as $n \to \infty$, there exists $m \in \mathbb{N}$ such that

$$\|\mathcal{K}_m - \mathcal{K}\|_{B(\mathcal{X},\mathcal{Y})} < \tfrac{1}{3}\,\epsilon\,.$$

Because $\mathcal{K}_m$ is compact, the set $\mathcal{K}_m B_{\mathcal{X}}$ is totally bounded. This implies there exists a finite set $S = \{u_j\}_{j=1}^k \subset B_{\mathcal{X}}$ such that for every $u \in B_{\mathcal{X}}$ there exists a $u_j \in S$ such that

$$\|\mathcal{K}_m u - \mathcal{K}_m u_j\|_{\mathcal{Y}} < \tfrac{1}{3}\,\epsilon\,.$$

For this u and u_j we see that

$$\begin{aligned}
\|\mathcal{K}u - \mathcal{K}u_j\|_{\mathcal{Y}} &\leq \|\mathcal{K}u - \mathcal{K}_m u\|_{\mathcal{Y}} + \|\mathcal{K}_m u - \mathcal{K}_m u_j\|_{\mathcal{Y}} + \|\mathcal{K}_m u_j - \mathcal{K}u_j\|_{\mathcal{Y}} \\
&\leq \|\mathcal{K}_m - \mathcal{K}\|_{B(\mathcal{X},\mathcal{Y})}\|u\|_{\mathcal{X}} + \|\mathcal{K}_m u - \mathcal{K}_m u_j\|_{\mathcal{Y}} + \|\mathcal{K}_m - \mathcal{K}\|_{B(\mathcal{X},\mathcal{Y})}\|u_j\|_{\mathcal{X}} \\
&< \tfrac{1}{3}\,\epsilon + \tfrac{1}{3}\,\epsilon + \tfrac{1}{3}\,\epsilon = \epsilon\,.
\end{aligned}$$

Hence, for every $u \in B_{\mathcal{X}}$ there exists a $u_j \in S$ such that $\|\mathcal{K}u - \mathcal{K}u_j\|_{\mathcal{Y}} < \epsilon$. Therefore the set $\mathcal{K}B_{\mathcal{X}}$ is totally bounded, whereby the operator $\mathcal{K}$ is compact. $\square$

The previous theorem would not be of much help unless there exists a sufficiently large class of compact linear operators with which to build the approximating sequences that it requires. This class is often provided by a class of *finite rank* operators.

Definition 1.1. *Let* $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ *and* $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ *be normed spaces. We say that* $\mathcal{K} \in B(\mathcal{X}, \mathcal{Y})$ *has* finite rank *if the range of* $\mathcal{K}$ *is a finite dimensional subspace of* $\mathcal{Y}$*, in which case the dimension of the range is called the* rank *of* $\mathcal{K}$*.*

The fact that finite rank operators are compact is a consequence of the Bolzano-Weierstrass Theorem, which implies that bounded subsets of Euclidean space are totally bounded.

2. Hölder-Young Bounds

None of the bounds presented below were actually derived by either Hölder or Young in the general setting given here. Rather, they generalize certain bounds first derived by them [2].

2.1. Lebesgue Spaces.

For any positive σ-finite measure space $(X, \Sigma_\mu, \mathrm{d}\mu)$ and any $p \in (0, \infty)$ we define the *Lebesgue space* $L^p(\mathrm{d}\mu)$ by

$$L^p(\mathrm{d}\mu) = \left\{ u \in M(\mathrm{d}\mu) \,:\, \int |u(x)|^p \,\mathrm{d}\mu(x) < \infty \right\} . \tag{2.1}$$

It is easy to check that $L^p(\mathrm{d}\mu)$ is a linear space [1]. For every $p \in (0, \infty)$ we define the magnitude of $u \in L^p(\mathrm{d}\mu)$ by

$$[u]_{L^p} = \left(\int |u(x)|^p \,\mathrm{d}\mu(x) \right)^{\frac{1}{p}} . \tag{2.2}$$

It is clear from (2.1) that for every $u \in M(\mathrm{d}\mu)$ we have $u \in L^p(\mathrm{d}\mu)$ if and only if $[u]_{L^p} < \infty$. It is also clear that $[\lambda\, u]_{L^p} = |\lambda|\, [u]_{L^p}$ for every $u \in L^p(\mathrm{d}\mu)$ and every $\lambda \in \mathbb{C}$. For every $p \in [1, \infty)$ the Minkowski inequality implies that $[\,\cdot\,]_{L^p}$ satisfies the triangle inequality, and is thereby a norm [1]. In that case $L^p(\mathrm{d}\mu)$ is a Banach space equipped with the norm

$$\|u\|_{L^p} = [u]_{L^p} = \left(\int |u(x)|^p \,\mathrm{d}\mu(x) \right)^{\frac{1}{p}} . \tag{2.3}$$

Moreover, for every $p \in (0, 1)$ one can show that $[\,\cdot\,]_{L^p}$ fails to satisfy the triangle inequality, and is thereby not a norm. However, in that case $L^p(\mathrm{d}\mu)$ is a Frechét space equipped with the metric

$$\mathrm{d}(u, v)_{L^p} = [u - v]_{L^p}^p = \int |u(x) - v(x)|^p \,\mathrm{d}\mu(x) . \tag{2.4}$$

Finally, we define the *Lebesgue space* $L^\infty(\mathrm{d}\mu)$ by

$$L^\infty(\mathrm{d}\mu) = \left\{ u \in M(\mathrm{d}\mu) \,:\, \operatorname*{ess\,sup}_{x \in X} \big\{ |u(x)| \big\} < \infty \right\} . \tag{2.5}$$

You can show that $L^\infty(\mathrm{d}\mu)$ is a Banach space equipped with the norm

$$\|u\|_{L^\infty} = \operatorname*{ess\,sup}_{x \in X} \big\{ |u(x)| \big\} = \inf \big\{ \alpha > 0 \,:\, \mu\big(E_u(\alpha)\big) = 0 \big\} , \tag{2.6}$$

where $E_u(\alpha) = \{x \in X \,:\, |u(x)| > \alpha\}$. Here we adopt the usual convention that $\inf\{\emptyset\} = \infty$.

2.1.1. *Hölder Inequalities.* For any positive σ-finite measure space $(X, \Sigma_\mu, \mathrm{d}\mu)$ the basic Hölder inequality goes as follows [1, 3]. Let $p, p^* \in [1, \infty]$ satisfy the duality relation

$$\frac{1}{p} + \frac{1}{p^*} = 1\,. \tag{2.7}$$

Then for every $u \in L^p(\mathrm{d}\mu)$ and $v \in L^{p^*}(\mathrm{d}\mu)$ we have $uv \in L^1(\mathrm{d}\mu)$ with

$$\int |u(x)\, v(x)|\, \mathrm{d}\mu(x) \leq \|u\|_{L^p}\, \|v\|_{L^{p^*}}\,. \tag{2.8}$$

The basic Hölder inequality has the following generalization. Let $p_1, p_2, \cdots, p_n, r \in [1, \infty]$ satisfy the relation

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} = \frac{1}{r}\,. \tag{2.9}$$

Then for every $u_1 \in L^{p_1}(\mathrm{d}\mu)$, $u_2 \in L^{p_2}(\mathrm{d}\mu)$, $\cdots$, $u_n \in L^{p_n}(\mathrm{d}\mu)$ we have $u_1\, u_2 \cdots u_n \in L^r(\mathrm{d}\mu)$ with

$$\|u_1\, u_2 \cdots u_n\|_{L^r} \leq \|u_1\|_{L^{p_1}}\, \|u_2\|_{L^{p_2}} \cdots \|u_n\|_{L^{p_n}}\,. \tag{2.10}$$

2.1.2. *L^p Riesz Representation Theorem.* For every p, $p^* \in [1, \infty]$ that satisfy the duality relation (2.7), the basic Hölder inequality (2.8) shows that every $v \in L^{p^*}$ defines a bounded linear functional $\ell_v \in (L^p)^* = B(L^p, \mathbb{C})$ by

$$\ell_v(u) = \int v(x)\, u(x)\, \mathrm{d}\mu(x)\,,$$

and that $\|\ell_v\|_{(L^p)^*} \leq \|v\|_{L^{p^*}}$, where by (1.3) we have

$$\|\ell_v\|_{(L^p)^*} = \sup_{u \in L^p} \left\{ \frac{|\ell_v(u)|}{\|u\|_{L^p}} \,:\, u \neq 0 \right\}. \tag{2.11}$$

We claim that $\|\ell_v\|_{(L^p)^*} = \|v\|_{L^{p^*}}$. This is clearly true when $v = 0$, so suppose that $\|v\|_{L^{p^*}} > 0$.

For $p \in (1, \infty]$ the argument is easy. One sees that $u = |v|^{p^*-1}\, \mathrm{sgn}(\overline{v}) \in L^p$ with

$$\|u\|_{L^p} = \|v\|_{L^{p^*}}^{p^*-1}\,, \qquad \text{and} \qquad \ell_v(u) = \|v\|_{L^{p^*}}^{p^*}\,.$$

Because $u \neq 0$, we infer from (2.11) that $\|\ell_v\|_{(L^p)^*} \geq \|v\|_{L^{p^*}}$.

For $p = 1$ we have $p^* = \infty$. Because $\|v\|_{L^\infty} > 0$, we see from (2.6) that for every α such that $0 < \alpha < \|v\|_{L^\infty}$ one has $\mu(\{x : |v(x)| > \alpha\}) > 0$. Let $E_\alpha \in \Sigma_\mu$ such that $E_\alpha \subset \{x : |v(x)| > \alpha\}$ and $0 < \mu(E_\alpha) < \infty$. One sees that $u_\alpha = \mathbf{1}_{E_\alpha}\, \mathrm{sgn}(\overline{v}) \in L^1$ with

$$\|u_\alpha\|_{L^1} = \mu(E_\alpha)\,, \qquad \text{and} \qquad \ell_v(u) = \int_{E_\alpha} |v(x)|\, \mathrm{d}\mu(x) \geq \alpha\, \mu(E_\alpha)\,.$$

Because $u_\alpha \neq 0$, we infer from (2.11) that $\|\ell_v\|_{(L^p)^*} \geq \alpha$. Because this holds for every α such that $0 < \alpha < \|v\|_{L^\infty}$, it follows that $\|\ell_v\|_{(L^p)^*} \geq \|v\|_{L^\infty}$.

Hence, for every $p \in [1, \infty]$ we have

$$\|\ell_v\|_{(L^p)^*} = \|v\|_{L^{p^*}}\,.$$

The mapping $v \mapsto \ell_v$ is therefore an isometry from $L^{p^*}(\mathrm{d}\mu)$ into $(L^p(\mathrm{d}\mu))^*$ for every $p \in [1, \infty]$. The L^p Riesz Representation Theorem asserts that this isometry is onto for every $p \in [1, \infty)$. It is a consequence of the Radon-Nikodym Theorem [1] or the L^p-Projection Theorem [3].

One of the most useful consequences of the L^p Riesz Representation Theorem is the following characterization of functions in $L^p(\mathrm{d}\mu)$.

Lemma 2.1. *Let $u \in M(\mathrm{d}\mu)$, $p \in [1, \infty]$, and $C \in [0, \infty)$. Then $u \in L^p(\mathrm{d}\mu)$ with $\|u\|_{L^p} \leq C$ if and only if*

$$\left| \int u(x)\, v(x)\, \mathrm{d}\mu(x) \right| \leq C\, \|v\|_{L^{p^*}} \quad \textit{for every } v \in L^{p^*}(\mathrm{d}\mu)\,. \tag{2.12}$$

Proof. The forward implication ($\Longrightarrow$) follows directly from the basic Hölder inequality (2.8). For $p = 1$ the other direction simply follows by taking $v = \mathrm{sgn}(\overline{u})$ in (2.12). For $p \in (1, \infty]$ define the linear functional ℓ_u by

$$\ell_u(v) = \int u(x)\, v(x)\, \mathrm{d}\mu(x) \quad \text{for every } v \in L^{p^*}(\mathrm{d}\mu)\,.$$

It follows from (2.12) that $\ell_u \in (L^{p^*}(\mathrm{d}\mu))^*$. Because $p^* \in [1, \infty)$ the L^p Riesz Representation Theorem then implies that there exists $w \in L^p(\mathrm{d}\mu)$ such that

$$\ell_u(v) = \int w(x)\, v(x)\, \mathrm{d}\mu(x) \quad \text{for every } v \in L^{p^*}(\mathrm{d}\mu)\,.$$

Hence, $u - w \in M(\mathrm{d}\mu)$ satisfies

$$\int \big(u(x) - w(x)\big)\, v(x)\, \mathrm{d}\mu(x) = 0 \quad \text{for every } v \in L^{p^*}(\mathrm{d}\mu)\,.$$

For every $E \in \Sigma_\mu$ such that $\mu(E) < \infty$ we have that $v = \mathbf{1}_E\, \mathrm{sgn}(\overline{u - w})$ is in $L^{p^*}(\mathrm{d}\mu)$ for every $p^* \in [1, \infty]$. The above condition therefore implies that $u - w \in M(\mathrm{d}\mu)$ satisfies

$$\int_E \big|u(x) - w(x)\big|\, \mathrm{d}\mu(x) = 0 \quad \text{for every } E \in \Sigma_\mu \text{ such that } \mu(E) < \infty\,.$$

We thereby conclude that $u = w \in L^p(\mathrm{d}\mu)$. It follows that $v = |u|^{p-1}\, \mathrm{sgn}(\overline{u}) \in L^{p^*}(\mathrm{d}\mu)$ with $\|v\|_{L^{p^*}} = \|u\|_{L^p}^{p-1}$. By then setting $v = |u|^{p-1}\, \mathrm{sgn}(\overline{u})$ into (2.12) we infer that $\|u\|_{L^p} \leq C$. $\square$

2.1.3. *Paired Bounds.* Let $\mathcal{K}^*$ denote formal adjoint of $\mathcal{K}$, which is given by

$$\mathcal{K}^* v(x) = \int \overline{k(x, y)}\, v(y)\, \mathrm{d}\nu(y)\,. \tag{2.13}$$

The operator $\mathcal{K}$ is bounded from $L^p(\mathrm{d}\mu)$ to $L^{q^*}(\mathrm{d}\nu)$ if and only if $\mathcal{K}^*$ is bounded from $L^q(\mathrm{d}\nu)$ to $L^{p^*}(\mathrm{d}\mu)$ where $p^*, q \in [1, \infty]$ are determined by the duality relations

$$\tfrac{1}{p} + \tfrac{1}{p^*} = 1\,, \quad \text{and} \quad \tfrac{1}{q} + \tfrac{1}{q^*} = 1\,. \tag{2.14}$$

Moreover, $\|\mathcal{K}^*\|_{B(L^q(\mathrm{d}\nu), L^{p^*}(\mathrm{d}\mu))} = \|\mathcal{K}\|_{B(L^p(\mathrm{d}\mu), L^{q^*}(\mathrm{d}\nu))}$.

We will use the following criterion to establish the boundedness of both $\mathcal{K}$ and $\mathcal{K}^*$.

Lemma 2.2. *Let $k \in M(\mathrm{d}\mu\, \mathrm{d}\nu)$ and $C \in [0, \infty)$ such that for every $u \in L^p(\mathrm{d}\mu)$ and $v \in L^q(\mathrm{d}\nu)$ we have*

$$\iint \big|k(x, y)\, u(x)\, \overline{v(y)}\big|\, \mathrm{d}\mu(x)\, \mathrm{d}\nu(y) \leq C\, \|u\|_{L^p} \|v\|_{L^q}\,. \tag{2.15}$$

Then $\mathcal{K} \in B(L^p(\mathrm{d}\mu), L^{q^*}(\mathrm{d}\nu))$ and $\mathcal{K}^* \in B(L^q(\mathrm{d}\nu), L^{p^*}(\mathrm{d}\mu))$ with

$$\|\mathcal{K}\|_{B(L^p, L^{q^*})} = \|\mathcal{K}^*\|_{B(L^q, L^{p^*})} \leq C\,. \tag{2.16}$$

Remark. The measures $\mathrm{d}\mu$ and $\mathrm{d}\nu$ will be dropped from the notation for norms as we did in (2.15) when there is no confusion about what measures are involved.

Proof. By (2.15) and the Fubini-Tonelli Theorem we see that for every $u \in L^p(\mathrm{d}\mu)$ and $v \in L^q(\mathrm{d}\nu)$ we have $\mathcal{K}u \in M(\mathrm{d}\nu)$, $\mathcal{K}^*v \in M(\mathrm{d}\mu)$, and

$$\int \overline{v(y)}\, \mathcal{K}u(y)\, \mathrm{d}\nu(y) = \iint k(x,y)\, u(x)\, \overline{v(y)}\, \mathrm{d}\mu(x)\, \mathrm{d}\nu(y) = \int u(x) \overline{\mathcal{K}^*v(x)}\, \mathrm{d}\mu(x)\,.$$

Hence, for every $u \in L^p(\mathrm{d}\mu)$ we have $\mathcal{K}u \in M(\mathrm{d}\nu)$ and

$$\left| \int \overline{v(y)}\, \mathcal{K}u(y)\, \mathrm{d}\nu(y) \right| \leq C\, \|u\|_{L^p} \|v\|_{L^q} \quad \text{for every } v \in L^q(\mathrm{d}\nu)\,,$$

By Lemma 2.1 we infer that $\mathcal{K}u \in L^{q^*}(\mathrm{d}\nu)$ and that $\|\mathcal{K}u\|_{L^{q^*}} \leq C\|u\|_{L^p}$. Because this holds for every $u \in L^p(\mathrm{d}\mu)$, we conclude that $\mathcal{K} \in B(L^p(\mathrm{d}\mu), L^{q^*}(\mathrm{d}\nu))$ and that $\|\mathcal{K}\|_{B(L^p,L^{q^*})} \leq C$.

Similarly, for every $v \in L^q(\mathrm{d}\nu)$ we have $\mathcal{K}^*v \in M(\mathrm{d}\mu)$ and

$$\left| \int \overline{u(x)}\, \mathcal{K}^*v(x)\, \mathrm{d}\mu(x) \right| \leq C\, \|u\|_{L^p} \|v\|_{L^q} \quad \text{for every } u \in L^p(\mathrm{d}\mu)\,,$$

By Lemma 2.1 we infer that $\mathcal{K}^*v \in L^{p^*}(\mathrm{d}\mu)$ and that $\|\mathcal{K}^*v\|_{L^{p^*}} \leq C\|v\|_{L^q}$. Because this holds for every $v \in L^q(\mathrm{d}\nu)$, we conclude that $\mathcal{K}^* \in B(L^q(\mathrm{d}\nu), L^{p^*}(\mathrm{d}\mu))$ and that $\|\mathcal{K}^*\|_{B(L^q,L^{p^*})} \leq C$. $\square$

2.2. **Iterated Norm Bounds.** We now give two basic bounds of the type (2.15).

Lemma 2.3. *Let $p, q \in [1,\infty]$. Let the kernel k satisfy the bound*

$$\|k\|_{L^{q^*}(\mathrm{d}\nu; L^{p^*}(\mathrm{d}\mu))} = \left(\left(\int |k(x,y)|^{p^*} \mathrm{d}\mu(x) \right)^{\frac{q^*}{p^*}} \mathrm{d}\nu(y) \right)^{\frac{1}{q^*}} < \infty\,. \tag{2.17}$$

Then for every $u \in L^p(\mathrm{d}\mu)$ and $v \in L^q(\mathrm{d}\nu)$ we have

$$\iint \left| k(x,y)\, u(x)\, \overline{v(y)} \right| \mathrm{d}\mu(x)\, \mathrm{d}\nu(y) \leq \|k\|_{L^{q^*}(L^{p^*})}\, \|u\|_{L^p}\, \|v\|_{L^q}\,. \tag{2.18}$$

Similarly, let the kernel k satisfy the bound

$$\|k\|_{L^{p^*}(\mathrm{d}\mu; L^{q^*}(\mathrm{d}\nu))} = \left(\left(\int |k(x,y)|^{q^*} \mathrm{d}\nu(y) \right)^{\frac{p^*}{q^*}} \mathrm{d}\mu(x) \right)^{\frac{1}{p^*}} < \infty\,. \tag{2.19}$$

Then for every $u \in L^p(\mathrm{d}\mu)$ and $v \in L^q(\mathrm{d}\nu)$ we have

$$\iint \left| k(x,y)\, u(x)\, \overline{v(y)} \right| \mathrm{d}\mu(x)\, \mathrm{d}\nu(y) \leq \|k\|_{L^{p^*}(L^{q^*})}\, \|u\|_{L^p}\, \|v\|_{L^q}\,. \tag{2.20}$$

Remark. The spaces $L^{q^*}(\mathrm{d}\nu; L^{p^*}(\mathrm{d}\mu))$ and $L^{p^*}(\mathrm{d}\mu; L^{q^*}(\mathrm{d}\nu))$ are called iterated spaces. They are equipped with the so-called iterated norms $\|\cdot\|_{L^{q^*}(\mathrm{d}\nu;L^{p^*}(\mathrm{d}\mu))}$ and $\|\cdot\|_{L^{p^*}(\mathrm{d}\mu;L^{q^*}(\mathrm{d}\nu))}$ defined above by (2.17) and (2.19). The bounds (2.18) and (2.20) are called iterated norm bounds.

Proof. Let $I(k,u,v)$ denote the quantity on the left-hand side of (2.18) and (2.20) — namely, let

$$I(k,u,v) = \iint \left| k(x,y)\, u(x)\, \overline{v(y)} \right| \mathrm{d}\mu(x)\, \mathrm{d}\nu(y)\,. \tag{2.21}$$

In order to ensure that this quantity makes sense, we assume for the moment that u, v, and k are simple functions with respect to the measures $\mathrm{d}\mu$, $\mathrm{d}\nu$, and $\mathrm{d}\mu\, \mathrm{d}\nu$ respectively.

The first iterated norm bound (2.18) is derived as follows. By the basic Hölder inequality (2.8) one has

$$\int \big|k(x,y)\, u(x)\big| \,\mathrm{d}\mu(x) \leq \left(\int |k(x,y)|^{p^*} \,\mathrm{d}\mu(x) \right)^{\frac{1}{p^*}} \|u\|_{L^p(\mathrm{d}\mu)} \,.$$

Upon first using this bound and then applying the Hölder inequality again, we derive the bound

$$\begin{aligned} I(k,u,v) &= \int \left(\int \big|k(x,y)\, u(x)\big| \,\mathrm{d}\mu(x) \right) |v(y)| \,\mathrm{d}\nu(y) \\ &\leq \int \left(\int |k(x,y)|^{p^*} \mathrm{d}\mu(x) \right)^{\frac{1}{p^*}} |v(y)| \,\mathrm{d}\nu(y)\, \|u\|_{L^p(\mathrm{d}\mu)} \\ &\leq \|k\|_{L^{q^*}(\mathrm{d}\nu; L^{p^*}(\mathrm{d}\mu))}\, \|u\|_{L^p(\mathrm{d}\mu)}\, \|v\|_{L^q(\mathrm{d}\nu)} \,. \end{aligned}$$

The first iterated norm bound (2.18) then follows by a density argument.

The second iterated norm bound (2.20) is derived by simply reversing the roles of x, u, p, and $\mathrm{d}\mu$ with those of y, v, q, and $\mathrm{d}\nu$. By the Hölder inequality one has

$$\int \big|k(x,y)\, \overline{v(y)}\big| \,\mathrm{d}\nu(y) \leq \left(\int |k(x,y)|^{q^*} \,\mathrm{d}\nu(y) \right)^{\frac{1}{q^*}} \|v\|_{L^q(\mathrm{d}\nu)} \,.$$

Upon first using this bound and then applying the Hölder inequality again, we obtain the bound

$$\begin{aligned} I(k,u,v) &= \int \left(\int \big|k(x,y)\, \overline{v(y)}\big| \,\mathrm{d}\nu(y) \right) |u(x)| \,\mathrm{d}\mu(x) \\ &\leq \int \left(\int |k(x,y)|^{q^*} \mathrm{d}\nu(y) \right)^{\frac{1}{q^*}} |u(x)| \,\mathrm{d}\mu(x)\, \|v\|_{L^q(\mathrm{d}\nu)} \\ &\leq \|k\|_{L^{p^*}(\mathrm{d}\mu; L^{q^*}(\mathrm{d}\nu))}\, \|u\|_{L^p(\mathrm{d}\mu)}\, \|v\|_{L^q(\mathrm{d}\nu)} \,. \end{aligned}$$

The second iterated norm bound (2.20) then follows by a density argument. □

Remark. The Minkowski inequality for integrals [1] implies that

$$\begin{aligned} &\|k\|_{L^{q^*}(\mathrm{d}\nu; L^{p^*}(\mathrm{d}\mu))} \leq \|k\|_{L^{p^*}(\mathrm{d}\mu; L^{q^*}(\mathrm{d}\nu))} \quad \text{whenever } p^* \leq q^* \,, \\ &\|k\|_{L^{p^*}(\mathrm{d}\mu; L^{q^*}(\mathrm{d}\nu))} \leq \|k\|_{L^{q^*}(\mathrm{d}\nu; L^{p^*}(\mathrm{d}\mu))} \quad \text{whenever } q^* \leq p^* \,. \end{aligned} \tag{2.22}$$

In the first case we can conclude that the first iterated norm bound (2.18) is the sharper one, whereby we conclude by Lemma 2.2 that $\mathcal{K} \in B(L^p, L^{q^*})$ and $\mathcal{K}^* \in B(L^q, L^{p^*})$ with

$$\|\mathcal{K}\|_{B(L^p, L^{q^*})} = \|\mathcal{K}^*\|_{B(L^q, L^{p^*})} \leq \|k\|_{L^{q^*}(L^{p^*})} \quad \text{for every } k \in L^{q^*}(\mathrm{d}\nu; L^{p^*}(\mathrm{d}\mu)) \,. \tag{2.23}$$

In the second case we can conclude that the second iterated norm bound (2.20) is the sharper one, whereby we conclude by Lemma 2.2 that $\mathcal{K} \in B(L^p, L^{q^*})$ and $\mathcal{K}^* \in B(L^q, L^{p^*})$ with

$$\|\mathcal{K}\|_{B(L^p, L^{q^*})} = \|\mathcal{K}^*\|_{B(L^q, L^{p^*})} \leq \|k\|_{L^{p^*}(L^{q^*})} \quad \text{for every } k \in L^{p^*}(\mathrm{d}\mu; L^{q^*}(\mathrm{d}\nu)) \,. \tag{2.24}$$

Remark: When either $1 \leq p^* \leq q^* < \infty$ and $k \in L^{q^*}(\mathrm{d}\nu; L^{p^*}(\mathrm{d}\mu))$ or $1 \leq q^* \leq p^* < \infty$ and $k \in L^{p^*}(\mathrm{d}\mu; L^{q^*}(\mathrm{d}\nu))$ then we can conclude that the bounded operators $\mathcal{K}$ and $\mathcal{K}^*$ from (2.23) and (2.24) are moreover compact. This is because one can show that the finite-rank kernels are dense in the spaces $L^{q^*}(\mathrm{d}\nu; L^{p^*}(\mathrm{d}\mu))$ and $L^{p^*}(\mathrm{d}\mu; L^{q^*}(\mathrm{d}\nu))$. The classical Hilbert-Schmidt compactness criterion is the special case $p = q = 2$.

Remark: When $p = q$ in the iterated spaces $L^{p^*}(\mathrm{d}\nu; L^{p^*}(\mathrm{d}\mu))$ and $L^{p^*}(\mathrm{d}\mu; L^{p^*}(\mathrm{d}\nu))$ coincide with

$$L^{p^*}(\mathrm{d}\nu; L^{p^*}(\mathrm{d}\mu)) = L^{p^*}(\mathrm{d}\mu; L^{p^*}(\mathrm{d}\nu)) = L^{p^*}(\mathrm{d}\mu\, \mathrm{d}\nu)\,.$$

Moreover, the iterated norms given by (2.17) and (2.19) also coincide with

$$\|k\|_{L^{p^*}(\mathrm{d}\nu; L^{p^*}(\mathrm{d}\mu))} = \|k\|_{L^{p^*}(\mathrm{d}\mu; L^{p^*}(\mathrm{d}\nu))} = \|k\|_{L^{p^*}(\mathrm{d}\mu\, \mathrm{d}\nu)}\,.$$

If these are finite then $\mathcal{K}$ is bounded from $L^p(\mathrm{d}\mu)$ to $L^{p^*}(\mathrm{d}\nu)$ and $\mathcal{K}^*$ is bounded from $L^p(\mathrm{d}\nu)$ to $L^{p^*}(\mathrm{d}\mu)$. If moreover $p^* < \infty$ then $\mathcal{K}$ and $\mathcal{K}^*$ are also compact by the previous remark.

Remark: In some cases the iterated norm bounds (2.18) and (2.20) are sharp. Specifically, it can be shown that when $p \in [1, \infty]$ and $q = 1$ one has

$$\|\mathcal{K}\|_{B(L^p, L^\infty)} = \|\mathcal{K}^*\|_{B(L^1, L^{p^*})} = \|k\|_{L^\infty(L^{p^*})} \quad \text{for every } k \in L^\infty(\mathrm{d}\nu; L^{p^*}(\mathrm{d}\mu))\,,$$

while when $p = 1$ and $q \in [1, \infty]$ one has

$$\|\mathcal{K}\|_{B(L^1, L^{q^*})} = \|\mathcal{K}^*\|_{B(L^q, L^\infty)} = \|k\|_{L^\infty(L^{q^*})} \quad \text{for every } k \in L^\infty(\mathrm{d}\mu; L^{q^*}(\mathrm{d}\nu))\,.$$

2.3. **Young Integral Operator Bounds.** The results of the previous section include the following. If $k \in L^\infty(\mathrm{d}\mu\, \mathrm{d}\nu)$ then for every $u \in L^1(\mathrm{d}\mu)$ and $v \in L^1(\mathrm{d}\nu)$ we have

$$\iint \left| k(x,y)\, u(x)\, \overline{v(y)} \right| \mathrm{d}\mu(x)\, \mathrm{d}\nu(y) \leq \|k\|_{L^\infty(\mathrm{d}\mu\, \mathrm{d}\nu)} \|u\|_{L^1}\, \|v\|_{L^1}\,. \tag{2.25}$$

If $k \in L^\infty(\mathrm{d}\mu; L^r(\mathrm{d}\nu))$ for some $r \in [1, \infty)$ then for every $u \in L^1(\mathrm{d}\mu)$ and $v \in L^{r^*}(\mathrm{d}\nu)$ we have

$$\iint \left| k(x,y)\, u(x)\, \overline{v(y)} \right| \mathrm{d}\mu(x)\, \mathrm{d}\nu(y) \leq \|k\|_{L^\infty(\mathrm{d}\mu; L^r(\mathrm{d}\nu))} \|u\|_{L^1}\, \|v\|_{L^{r^*}}\,. \tag{2.26}$$

If $k \in L^\infty(\mathrm{d}\nu; L^r(\mathrm{d}\mu))$ for some $r \in [1, \infty)$ then for every $u \in L^{r^*}(\mathrm{d}\mu)$ and $v \in L^1(\mathrm{d}\nu)$ we have

$$\iint \left| k(x,y)\, u(x)\, \overline{v(y)} \right| \mathrm{d}\mu(x)\, \mathrm{d}\nu(y) \leq \|k\|_{L^\infty(\mathrm{d}\nu; L^r(\mathrm{d}\mu))} \|u\|_{L^{r^*}}\, \|v\|_{L^1}\,. \tag{2.27}$$

In this section we show that if $k \in L^\infty(L^r)(\mathrm{d}\mu, \mathrm{d}\nu) = L^\infty(\mathrm{d}\nu; L^r(\mathrm{d}\mu)) \cap L^\infty(\mathrm{d}\mu; L^r(\mathrm{d}\nu))$ for some $r \in [1, \infty)$ then, in addition to the bounds (2.26) and (2.27), we have an entire family of Young integral operator bounds.

Theorem 2.1. *Let $k \in L^\infty(L^r)(\mathrm{d}\mu, \mathrm{d}\nu)$ for some $r \in [1, \infty)$. Let $p, q \in [1, r^*]$ satisfy the relation*

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2\,. \tag{2.28}$$

Then for every $u \in L^p(\mathrm{d}\mu)$ and $v \in L^q(\mathrm{d}\nu)$ we have

$$\iint \left| k(x,y)\, u(x)\, \overline{v(y)} \right| \mathrm{d}\mu(x)\, \mathrm{d}\nu(y) \leq \|k\|_{L^\infty(\mathrm{d}\mu; L^r(\mathrm{d}\nu))}^{\frac{r}{p^*}} \|k\|_{L^\infty(\mathrm{d}\nu; L^r(\mathrm{d}\mu))}^{\frac{r}{q^*}} \|u\|_{L^p}\, \|v\|_{L^q}\,. \tag{2.29}$$

Moreover, we have $\mathcal{K} \in B(L^p(\mathrm{d}\mu), L^{q^*}(\mathrm{d}\nu))$ and $\mathcal{K}^* \in B(L^q(\mathrm{d}\nu), L^{p^*}(\mathrm{d}\mu))$ with

$$\|\mathcal{K}\|_{B(L^p, L^{q^*})} = \|\mathcal{K}^*\|_{B(L^q, L^{p^*})} \leq \|k\|_{L^\infty(\mathrm{d}\mu; L^r(\mathrm{d}\nu))}^{\frac{r}{p^*}}\, \|k\|_{L^\infty(\mathrm{d}\nu; L^r(\mathrm{d}\mu))}^{\frac{r}{q^*}}\,. \tag{2.30}$$

Remark. The case $r = \infty$ is already covered by (2.25) because in that case relation (2.28) would require that $p = q = 1$. The case $r \in [1, \infty)$ and $p = 1$ is already covered by (2.26) because in that case relation (2.28) would require that $q = r^*$. The case $r \in [1, \infty)$ and $q = 1$ is already covered by (2.27) because in that case relation (2.28) would require that $p = r^*$.

Proof. The case $q = \infty$ is covered by (2.26) with $r = 1$ because in that case relation (2.28) would require that $p = r = 1$. The case $p = \infty$ is covered by (2.27) with $r = 1$ because in that case relation (2.28) would require that $q = r = 1$. Therefore we only have to establish the case when $p, q, r \in [1, \infty)$ satisfy relation (2.28).

We will apply the general Hölder inequality (2.10) to the three functions

$$U(x,y) = |k(x,y)|^{\frac{r}{p^*}} |v(y)|^{\frac{q}{p^*}} \,, \;\; V(x,y) = |k(x,y)|^{\frac{r}{q^*}} |u(x)|^{\frac{p}{q^*}} \,, \;\; W(x,y) = |u(x)|^{\frac{p}{r^*}} |v(y)|^{\frac{q}{r^*}} \,.$$

Relation (2.28) implies that

$$\tfrac{r}{p^*} + \tfrac{r}{q^*} = 1 \,, \qquad \tfrac{q}{p^*} + \tfrac{q}{r^*} = 1 \,, \qquad \tfrac{p}{q^*} + \tfrac{p}{r^*} = 1 \,.$$

One thereby sees that

$$|k(x,y)| \, |u(x)| \, |v(y)| = U(x,y) \, V(x,y) \, W(x,y) \,.$$

Because relation (2.28) also implies that $\frac{1}{p^*} + \frac{1}{q^*} + \frac{1}{r^*} = 1$, the general Hölder inequality (2.10) yields

$$\begin{aligned} I(u,v,w) &= \iint |k(x,y)| \, |u(x)| \, |v(y)| \, \mathrm{d}\mu(x) \, \mathrm{d}\nu(y) \\ &= \iint U(x,y) \, V(x,y) \, W(x,y) \, \mathrm{d}\mu(x) \, \mathrm{d}\nu(y) \\ &\leq \|U\|_{L^{p^*}(\mathrm{d}\mu \, \mathrm{d}\nu)} \|V\|_{L^{q^*}(\mathrm{d}\mu \, \mathrm{d}\nu)} \|W\|_{L^{r^*}(\mathrm{d}\mu \, \mathrm{d}\nu)} \\ &= \|k\|_{L^\infty(\mathrm{d}\mu; L^r(\mathrm{d}\nu))}^{\frac{r}{p^*}} \|v\|_{L^q}^{\frac{q}{p^*}} \cdot \|k\|_{L^\infty(\mathrm{d}\nu; L^r(\mathrm{d}\mu))}^{\frac{r}{q^*}} \|u\|_{L^p}^{\frac{p}{q^*}} \cdot \|u\|_{L^p}^{\frac{p}{r^*}} \|v\|_{L^q}^{\frac{q}{r^*}} \\ &= \|k\|_{L^\infty(\mathrm{d}\mu; L^r(\mathrm{d}\nu))}^{\frac{r}{p^*}} \, \|k\|_{L^\infty(\mathrm{d}\nu; L^r(\mathrm{d}\mu))}^{\frac{r}{q^*}} \, \|u\|_{L^p} \|v\|_{L^q} \,, \end{aligned}$$

whereby the Young integral operator bound (2.29) holds. Assertion (2.30) then follows from Lemma 2.2. □

2.4. **Interpolation Bounds.** The family of Young integral bounds (2.29) belongs to the larger class of interpolation bounds. We will develop interpolation bounds in the following setting. Suppose that for some $p_0, q_0, p_1, q_1 \in [1, \infty]$ the kernel k satisfies the bounds

$$\begin{aligned} &\iint \left| k(x,y) \, u(x) \, \overline{v(y)} \right| \mathrm{d}\mu(x) \, \mathrm{d}\nu(y) \leq C_0 \, \|u\|_{L^{p_0}} \|v\|_{L^{q_0}} \\ &\qquad \text{for every } u \in L^{p_0}(\mathrm{d}\mu) \text{ and } v \in L^{q_0}(\mathrm{d}\nu) \,, \\ &\iint \left| k(x,y) \, u(x) \, \overline{v(y)} \right| \mathrm{d}\mu(x) \, \mathrm{d}\nu(y) \leq C_1 \, \|u\|_{L^{p_1}} \|v\|_{L^{q_1}} \\ &\qquad \text{for every } u \in L^{p_1}(\mathrm{d}\mu) \text{ and } v \in L^{q_1}(\mathrm{d}\nu) \,, \end{aligned} \tag{2.31}$$

These bounds imply the operator $\mathcal{K}$ belongs to $B(L^{p_0}(\mathrm{d}\mu), L^{q_0^*}(\mathrm{d}\nu))$ and to $B(L^{p_1}(\mathrm{d}\mu), L^{q_1^*}(\mathrm{d}\nu))$, where the usual duality relations $\frac{1}{q_0} + \frac{1}{q_0^*} = 1$ and $\frac{1}{q_1} + \frac{1}{q_1^*} = 1$ hold. Interpolation will allow us to extend all of these results to other spaces.

Rather than develop the full Riesz-Thorin interpolation theory for L^p spaces [1], here we will simply employ the following elementary interpolation lemma.

Lemma 2.4. *If the kernel k satisfies the bounds (2.31) for some $p_0, q_0, p_1, q_1 \in [1, \infty]$ then for every $t \in [1, \infty]$ it satisfies the interpolation bound*

$$\iint \left| k(x,y)\, u(x)\, \overline{v(y)} \right| \mathrm{d}\mu(x)\, \mathrm{d}\nu(y) \leq C_0^{\frac{1}{t^*}} C_1^{\frac{1}{t}} \, \|u\|_{L^p} \|v\|_{L^q} \quad \text{for every } u \in L^p(\mathrm{d}\mu) \text{ and } v \in L^q(\mathrm{d}\nu)\,, \tag{2.32}$$

where $t^* \in [1, \infty]$ satisfies $\frac{1}{t} + \frac{1}{t^*} = 1$, and p, $q \in [1, \infty]$ satisfy the interpolation relations

$$\frac{1}{p} = \frac{1}{t^* p_0} + \frac{1}{t p_1}\,, \qquad \frac{1}{q} = \frac{1}{t^* q_0} + \frac{1}{t q_1}\,. \tag{2.33}$$

Moreover, we have $\mathcal{K} \in B(L^p(\mathrm{d}\mu), L^{q^*}(\mathrm{d}\nu))$ and $\mathcal{K}^* \in B(L^q(\mathrm{d}\nu), L^{p^*}(\mathrm{d}\mu))$ with

$$\|\mathcal{K}\|_{B(L^p, L^{q^*})} = \|\mathcal{K}^*\|_{B(L^q, L^{p^*})} \leq C_0^{\frac{1}{t^*}} C_1^{\frac{1}{t}}\,. \tag{2.34}$$

Here the usual duality relations $\frac{1}{p} + \frac{1}{p^*} = 1$, and $\frac{1}{q} + \frac{1}{q^*} = 1$ hold.

Proof. Suppose that either $p_0 \neq p_1$ or $q_0 \neq q_1$, because otherwise there is nothing to prove. When $t = \infty$ or $t = 1$ then (2.32) reduces to the first or second bound of (2.31) respectively. We thereby only need to establish (2.32) when $t \in (1, \infty)$. Because (2.32) clearly holds when either $u = 0$ or $v = 0$, we only need to consider cases when both $u \neq 0$ and $v \neq 0$.

We first consider the case when neither $p_0 = p_1 = \infty$ nor $q_0 = q_1 = \infty$. Because $t \in (1, \infty)$ we see from (2.33) that p, $q \in [1, \infty)$. Hence, whenever $u \in L^p(\mathrm{d}\mu)$ and $v \in L^q(\mathrm{d}\nu)$ we observe that $|u|^{\frac{p}{p_0}} \in L^{p_0}(\mathrm{d}\mu)$, $|u|^{\frac{p}{p_1}} \in L^{p_1}(\mathrm{d}\mu)$, $|v|^{\frac{q}{q_0}} \in L^{q_0}(\mathrm{d}\nu)$, and $|v|^{\frac{q}{q_1}} \in L^{q_1}(\mathrm{d}\nu)$ with

$$\begin{aligned} \big\| |u|^{\frac{p}{p_0}} \big\|_{L^{p_0}} &= \|u\|_{L^p}^{\frac{p}{p_0}}\,, & \big\| |v|^{\frac{q}{q_0}} \big\|_{L^{q_0}} &= \|v\|_{L^q}^{\frac{q}{q_0}}\,, \\ \big\| |u|^{\frac{p}{p_1}} \big\|_{L^{p_1}} &= \|u\|_{L^p}^{\frac{p}{p_1}}\,, & \big\| |v|^{\frac{q}{q_1}} \big\|_{L^{q_1}} &= \|v\|_{L^q}^{\frac{q}{q_1}}\,. \end{aligned} \tag{2.35}$$

Given (2.33), for every $\lambda \in (0, \infty)$ the classical Young's inequality gives

$$|u|\,|v| = |u|^{\frac{p}{t^* p_0}}\,|v|^{\frac{q}{t^* q_0}}\,|u|^{\frac{p}{t p_1}}\,|v|^{\frac{q}{t q_1}} \leq \frac{\lambda^{t^*}}{t^*}\,|u|^{\frac{p}{p_0}}\,|v|^{\frac{q}{q_0}} + \frac{1}{t\lambda^t}\,|u|^{\frac{p}{p_1}}\,|v|^{\frac{q}{q_1}}\,.$$

Upon multiplying this inequality by $|k|$, integrating with respect to $\mathrm{d}\mu\,\mathrm{d}\nu$ over $X \times Y$, and using the assumed bounds (2.31) along with the observations (2.35), we find that for every $u \in L^p(\mathrm{d}\mu)$, every $v \in L^q(\mathrm{d}\nu)$, and every $\lambda \in (0, \infty)$ we have the bound

$$\iint \left| k(x,y)\, u(x)\, \overline{v(y)} \right| \mathrm{d}\mu(x)\, \mathrm{d}\nu(y) \leq \frac{\lambda^{t^*}}{t^*}\, C_0\, \|u\|_{L^p}^{\frac{p}{p_0}} \|v\|_{L^q}^{\frac{q}{q_0}} + \frac{1}{t\lambda^t}\, C_1\, \|u\|_{L^p}^{\frac{p}{p_1}} \|v\|_{L^q}^{\frac{q}{q_1}}\,. \tag{2.36}$$

When $u \neq 0$ and $v \neq 0$ the right-hand side above attains its minimum over $\lambda \in (0, \infty)$ at

$$\lambda = \left(\frac{C_1\, \|u\|_{L^p}^{\frac{p}{p_1}} \|v\|_{L^q}^{\frac{q}{q_1}}}{C_0\, \|u\|_{L^p}^{\frac{p}{p_0}} \|v\|_{L^q}^{\frac{q}{q_0}}} \right)^{\frac{1}{t^* t}}\,.$$

The interpolation bound (2.32) is then obtained by setting this value of λ into (2.36).

The cases when either $p_0 = p_1 = \infty$ or $q_0 = q_1 = \infty$ can be treated in the same framework. When $p_0 = p_1 = \infty$ we can set $p = \infty$ and $\frac{p}{p_0} = \frac{p}{p_0} = 1$ in the above argument and it goes through as written. Similarly, when $q_0 = q_1 = \infty$ we can set $q = \infty$ and $\frac{q}{q_0} = \frac{q}{q_0} = 1$ in the above argument and it goes through as written. We have therefore established the interpolation bound (2.32) for all cases. Assertion (2.34) then follows from Lemma 2.2. $\square$

We now apply the Interpolation Lemma 2.4 to a kernel k which for some $r, s \in [1, \infty]$ satisfies the bounds

$$\begin{aligned} \|k\|_{L^s(\mathrm{d}\nu;L^r(\mathrm{d}\mu))} &= \left(\int \left(\int |k(x,y)|^r \mathrm{d}\mu(x) \right)^{\frac{s}{r}} \mathrm{d}\nu(y) \right)^{\frac{1}{s}} < \infty \,, \\ \|k\|_{L^s(\mathrm{d}\mu;L^r(\mathrm{d}\nu))} &= \left(\int \left(\int |k(x,y)|^r \mathrm{d}\nu(y) \right)^{\frac{s}{r}} \mathrm{d}\mu(x) \right)^{\frac{1}{s}} < \infty \,. \end{aligned} \tag{2.37}$$

Without loss of generality we can assume $r \leq s$ because in that case $\|k\|_{L^s(L^r)} \leq \|k\|_{L^r(L^s)}$ for each of the above norms. We can assume moreover that $r < s$ because when $r = s$ the bounds in (2.37) coincide, so the Interpolation Lemma cannot yield further boundedness results.

Remark: In the symmetric setting in which $(X, \Sigma_\mu, \mathrm{d}\mu) = (Y, \Sigma_\nu, \mathrm{d}\nu)$ and $|k(y,x)| = |k(x,y)|$, the two bounds in (2.37) reduce to the single bound

$$\|k\|_{L^s(\mathrm{d}\mu;L^r(\mathrm{d}\mu))} = \left(\int \left(\int |k(x,y)|^r \mathrm{d}\mu(x) \right)^{\frac{s}{r}} \mathrm{d}\mu(y) \right)^{\frac{1}{s}} < \infty \,. \tag{2.38}$$

This setting is common in applications.

The iterated norm bounds of Section 2.2 show that the bounds (2.37) on k imply the following. Because $k \in L^s(\mathrm{d}\nu; L^r(\mathrm{d}\mu))$, then for every $u \in L^{r^*}(\mathrm{d}\mu)$ and $v \in L^{s^*}(\mathrm{d}\nu)$ we have

$$\iint \left| k(x,y)\, u(x)\, \overline{v(y)} \right| \mathrm{d}\mu(x)\, \mathrm{d}\nu(y) \leq \|k\|_{L^s(\mathrm{d}\nu;(L^r(\mathrm{d}\mu))} \|u\|_{L^{r^*}} \|v\|_{L^{s^*}} \,. \tag{2.39}$$

Because $k \in L^s(\mathrm{d}\mu; L^r(\mathrm{d}\nu))$, then for every $u \in L^{s^*}(\mathrm{d}\mu)$ and $v \in L^{r^*}(\mathrm{d}\nu)$ we have

$$\iint \left| k(x,y)\, u(x)\, \overline{v(y)} \right| \mathrm{d}\mu(x)\, \mathrm{d}\nu(y) \leq \|k\|_{L^s(\mathrm{d}\mu;(L^r(\mathrm{d}\nu))} \|u\|_{L^{s^*}} \|v\|_{L^{r^*}} \,. \tag{2.40}$$

In this section we show that because $k \in L^s(L^r)(\mathrm{d}\mu, \mathrm{d}\nu) = L^s(\mathrm{d}\nu; L^r(\mathrm{d}\mu)) \cap L^s(\mathrm{d}\mu; L^r(\mathrm{d}\nu))$ then, in addition to the bounds (2.39) and (2.40), we have a family of interpolation bounds.

Theorem 2.2. *Let $k \in L^s(L^r)(\mathrm{d}\mu\, \mathrm{d}\nu)$ for some $r, s \in [1, \infty]$ such that $r < s$. Let $p, q \in [s^*, r^*]$ satisfy the relation*

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{s} = 2 \,. \tag{2.41}$$

Then for every $u \in L^p(\mathrm{d}\mu)$ and $v \in L^q(\mathrm{d}\nu)$ we have the interpolation bound

$$\iint \left| k(x,y)\, u(x)\, \overline{v(y)} \right| \mathrm{d}\mu(x)\, \mathrm{d}\nu(y) \leq \|k\|^{\frac{1}{t^*}}_{L^s(\mathrm{d}\nu;L^r(\mathrm{d}\mu))} \|k\|^{\frac{1}{t}}_{L^s(\mathrm{d}\mu;L^r(\mathrm{d}\nu))} \|u\|_{L^p} \|v\|_{L^q} \,, \tag{2.42}$$

where t^* and t are given by

$$\frac{1}{t^*} = \frac{\dfrac{1}{s^*} - \dfrac{1}{p}}{\dfrac{1}{s^*} - \dfrac{1}{r^*}} = \frac{\dfrac{1}{q} - \dfrac{1}{r^*}}{\dfrac{1}{s^*} - \dfrac{1}{r^*}} \,, \qquad \frac{1}{t} = \frac{\dfrac{1}{p} - \dfrac{1}{r^*}}{\dfrac{1}{s^*} - \dfrac{1}{r^*}} = \frac{\dfrac{1}{s^*} - \dfrac{1}{q}}{\dfrac{1}{s^*} - \dfrac{1}{r^*}} \,. \tag{2.43}$$

Moreover, we have $\mathcal{K} \in B(L^p(\mathrm{d}\mu), L^{q^*}(\mathrm{d}\nu))$ and $\mathcal{K}^* \in B(L^q(\mathrm{d}\nu), L^{p^*}(\mathrm{d}\mu))$ with

$$\|\mathcal{K}\|_{B(L^p,L^{q^*})} = \|\mathcal{K}^*\|_{B(L^q,L^{p^*})} \leq \|k\|^{\frac{1}{t^*}}_{L^s(\mathrm{d}\nu;L^r(\mathrm{d}\mu))} \|k\|^{\frac{1}{t}}_{L^s(\mathrm{d}\mu;L^r(\mathrm{d}\nu))} \,. \tag{2.44}$$

When $s < \infty$ the operators $\mathcal{K}$ and $\mathcal{K}^*$ are also compact.

Remark: When $s = \infty$ this reduces to the Young Integral Operator Theorem 2.1.

Proof. Upon applying the Interpolation Lemma 2.4 to the bounds (2.39) and (2.40) with $p_0 = r^*$, $q_0 = s^*$, $p_1 = s^*$, $q_1 = r^*$, $C_0 = \|k\|_{L^s(\mathrm{d}\nu;L^r(\mathrm{d}\mu))}$, and $C_1 = \|k\|_{L^s(\mathrm{d}\mu;L^r(\mathrm{d}\nu))}$, for every $t \in [1, \infty]$ we obtain the interpolation bound

$$\iint \big|k(x,y)\,u(x)\,\overline{v(y)}\big|\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y) \leq \|k\|^{\frac{1}{t^*}}_{L^s(\mathrm{d}\nu;L^r(\mathrm{d}\mu))}\|k\|^{\frac{1}{t}}_{L^s(\mathrm{d}\mu;L^r(\mathrm{d}\nu))}\|u\|_{L^p}\|v\|_{L^q} \quad \text{for every } u \in L^p(\mathrm{d}\mu) \text{ and } v \in L^q(\mathrm{d}\nu)\,, \tag{2.45}$$

where

$$\frac{1}{p} = \frac{1}{t^* r^*} + \frac{1}{ts^*}\,, \qquad \frac{1}{q} = \frac{1}{t^* s^*} + \frac{1}{tr^*}\,. \tag{2.46}$$

It is clear from (2.46) that $p, q \in [s^*, r^*]$ and that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r^*} + \frac{1}{s^*}\,. \tag{2.47}$$

The relation is equivalent to relation (2.41).

Conversely, if $p, q \in [s^*, r^*]$ and relation (2.47) holds then there exists a unique $t \in [1, \infty]$ such that p and q are given by (2.46) — namely, the unique t given by (2.43). Hence, bound (2.45) is exactly bound (2.42). Assertion (2.44) then follows from Lemma 2.2.

Finally, the fact that the operators $\mathcal{K}$ and $\mathcal{K}^*$ are compact when $s < \infty$ follows because in that case one can show that the finite-rank kernels are dense in the spaces $L^s(\mathrm{d}\nu; L^r(\mathrm{d}\mu))$ and $L^s(\mathrm{d}\mu; L^r(\mathrm{d}\nu))$. □

Remark: When $r \in [1, 2]$ (so that $r \leq r^*$) and $s = r^*$ then for every $p \in [r, r^*]$ one sees that $\mathcal{K} \in B(L^p(\mathrm{d}\mu), L^p(\mathrm{d}\nu))$ and $\mathcal{K}^* \in B(L^{p^*}(\mathrm{d}\nu), L^{p^*}(\mathrm{d}\mu))$ with

$$\|\mathcal{K}\|_{B(L^p,L^p)} = \|\mathcal{K}^*\|_{B(L^{p^*},L^{p^*})} \leq \|k\|^{\frac{1}{t^*}}_{L^{r^*}(\mathrm{d}\nu;L^r(\mathrm{d}\mu))}\|k\|^{\frac{1}{t}}_{L^{r^*}(\mathrm{d}\mu;L^r(\mathrm{d}\nu))}\,,$$

where t is given by (2.43). In this case $q = p^*$.

Remark: Let p satisfy $\frac{2}{p} = \frac{1}{r^*} + \frac{1}{s^*}$. Notice that p is the harmonic mean of r^* and s^*, so that $p \in [s^*, r^*]$. One sees that $\mathcal{K} \in B(L^p(\mathrm{d}\mu), L^{p^*}(\mathrm{d}\nu))$ and $\mathcal{K}^* \in B(L^p(\mathrm{d}\nu), L^{p^*}(\mathrm{d}\mu))$ with

$$\|\mathcal{K}\|_{B(L^p,L^{p^*})} = \|\mathcal{K}^*\|_{B(L^p,L^{p^*})} \leq \|k\|^{\frac{1}{2}}_{L^s(\mathrm{d}\nu;L^r(\mathrm{d}\mu))}\|k\|^{\frac{1}{2}}_{L^s(\mathrm{d}\mu;L^r(\mathrm{d}\nu))}\,.$$

In this case $q = p$.

3. Hardy-Littlewood Bounds

The interpolation bound (2.42) cannot be applied to kernels over $\mathbb{R}^D \times \mathbb{R}^D$ of the form $k(x, y) = |x - y|^{-\frac{D}{r}}$ for some $r \in (1, \infty)$ when $\mathrm{d}\mu$ and $\mathrm{d}\nu$ are each Lebesgue measure because in that case both $\|k\|_{L^s(\mathrm{d}\nu;L^r(\mathrm{d}\mu))}$ and $\|k\|_{L^s(\mathrm{d}\mu;L^r(\mathrm{d}\nu))}$ are not finite. This problem was overcome by bounds that grew out of the pioneering work of Hardy and Littlewood [2]. Their work led to a class of spaces that allow the treatment of such kernels — namely, the weak Lebesgue spaces.

3.1. **Weak Lebesgue Spaces.** For any positive σ-finite measure space $(X, \Sigma_\mu, \mathrm{d}\mu)$ and any $p \in (0, \infty)$ we define the *weak Lebesgue space* $L^p_w(\mathrm{d}\mu)$ by

$$L^p_w(\mathrm{d}\mu) = \left\{ u \in M(\mathrm{d}\mu) \, : \, \sup_{\alpha>0} \left\{ \alpha^p \mu(E_u(\alpha)) \right\} < \infty \right\} , \tag{3.1}$$

where $E_u(\alpha) = \{x \in X \, : \, |u(x)| > \alpha\}$. It is easy to check that $L^p_w(\mathrm{d}\mu)$ is a linear space.

For every $p \in (0, \infty)$ it is clear that $L^p(\mathrm{d}\mu) \subset L^p_w(\mathrm{d}\mu)$. Indeed, for every $u \in L^p(\mathrm{d}\mu)$ and every $\alpha > 0$ the Chebyshev inequality yields

$$\mu(E_u(\alpha)) = \int_{E_u(\alpha)} \mathrm{d}\mu(x) \leq \frac{1}{\alpha^p} \int_{E_u(\alpha)} |u(x)|^p \, \mathrm{d}\mu(x) \leq \frac{1}{\alpha^p} \int |u(x)|^p \, \mathrm{d}\mu(x) = \frac{1}{\alpha^p} \, [u]^p_{L^p(\mathrm{d}\mu)} \, .$$

It thereby follows that

$$\sup_{\alpha>0} \left\{ \alpha^p \mu(E_u(\alpha)) \right\} \leq [u]^p_{L^p(\mathrm{d}\mu)} < \infty \, ,$$

whereby $u \in L^p_w(\mathrm{d}\mu)$. In general $L^p_w(\mathrm{d}\mu)$ is larger than $L^p(\mathrm{d}\mu)$. For example, when $X = \mathbb{R}^D$ and $\mathrm{d}\mu$ is the unsual Lebesgue measure on $\mathbb{R}^D$ then it can be shown that the function $u(x) = |x|^{-\frac{D}{p}}$ is in $L^p_w(\mathrm{d}\mu)$ but it is clearly not in $L^p(\mathrm{d}\mu)$.

For every $p \in (0, \infty)$ we define the magnitude of every $u \in L^p_w(\mathrm{d}\mu)$ by

$$[u]_{L^p_w} = \left(\sup_{\alpha>0} \left\{ \alpha^p \mu(E_u(\alpha)) \right\} \right)^{\frac{1}{p}} \, . \tag{3.2}$$

It is clear from (3.1) that $u \in L^p_w(\mathrm{d}\mu)$ if and only if $[u]_{L^p_w} < \infty$. However, $[\,\cdot\,]_{L^p_w}$ is not a norm. While it satisfies $[\lambda u]_{L^p_w} = |\lambda| \, [u]_{L^p_w}$ for every $u \in L^p_w(\mathrm{d}\mu)$ and $\lambda \in \mathbb{C}$, it fails to satisfy the triangle inequality. However, the next result shows there is an equivalent norm for $p \in (1, \infty)$.

Theorem 3.1. *For every $p \in (1, \infty)$ and every $u \in M(\mathrm{d}\mu)$ we define*

$$\|u\|_{L^p_w} = \sup_{E \in \Sigma_\mu} \left\{ \frac{1}{\mu(E)^{\frac{1}{p^*}}} \int_E |u(x)| \, \mathrm{d}\mu(x) \, : \, \mu(E) \in (0, \infty) \right\} . \tag{3.3}$$

For every $u \in M(\mathrm{d}\mu)$ we can show that $u \in L^p_w(\mathrm{d}\mu)$ if and only if $\|u\|_{L^p_w} < \infty$. Moreover,

$$[u]_{L^p_w} \leq \|u\|_{L^p_w} \leq p^* [u]_{L^p_w} \quad \textit{for every } u \in L^p_w(\mathrm{d}\mu) \, . \tag{3.4}$$

Remark. It is easily checked from definition (3.3) that $\| \cdot \|_{L^p_w}$ is a norm. The result stated above and proved below shows that the space $L^p_w(\mathrm{d}\mu)$ is characterized by the finiteness of this norm for every $p \in (1, \infty)$. Finally, if $u \in L^p(\mathrm{d}\mu)$ for some $p \in (1, \infty)$ then by applying the Hölder inequality inside the supremum of (3.3) and using the fact that $\|\mathbf{1}_E\|_{L^{p^*}} = \mu(E)^{\frac{1}{p^*}}$ shows that $\|u\|_{L^p_w} \leq \|u\|_{L^p}$. Here $\mathbf{1}_E$ denotes the indicator function of the set E.

Proof. First assume that $\|u\|_{L^p_w} < \infty$. We claim this implies $\mu(E_u(\alpha)) < \infty$ for every $\alpha > 0$. Indeed, suppose otherwise. Then $\mu(E_u(\alpha)) = \infty$ for some $\alpha > 0$. One can then construct a sequence $\{E_n\}_{n\in\mathbb{N}} \subset \Sigma_\mu$ such that $E_n \subset E_u(\alpha) = \{x \in X \, : \, |u(x)| > \alpha\}$ and $\mu(E_n) \in (n, \infty)$ for every $n \in \mathbb{N}$. It follows that

$$\frac{1}{\mu(E_n)^{\frac{1}{p^*}}} \int_{E_n} |u(x)| \, \mathrm{d}\mu(x) \geq \frac{1}{\mu(E_n)^{\frac{1}{p^*}}} \, \mu(E_n) \, \alpha = \mu(E_n)^{\frac{1}{p}} \, \alpha \to \infty \quad \text{as } n \to \infty \, .$$

But by (3.3) this contradicts $\|u\|_{L^p_w} < \infty$. Hence, $\mu(E_u(\alpha)) < \infty$ for every $\alpha > 0$. Moreover, whenever $\mu(E_u(\alpha)) > 0$ we have $\mu(E_u(\alpha)) \in (0,\infty)$ and by (3.3)

$$\mu(E_u(\alpha)) = \int_{E_u(\alpha)} \mathrm{d}\mu(x) \leq \frac{1}{\alpha} \int_{E_u(\alpha)} |u(x)| \, \mathrm{d}\mu(x) \leq \frac{1}{\alpha} \|u\|_{L^p_w} \, \mu(E_u(\alpha))^{\frac{1}{p^*}} .$$

This implies that $\mu(E_u(\alpha))^{\frac{1}{p}} \leq \|u\|_{L^p_w}/\alpha$ for every $\alpha > 0$. It thereby follows from (3.2) that

$$[u]_{L^p_w} = \left(\sup_{\alpha>0} \big\{ \alpha^p \mu(E_u(\alpha)) \big\} \right)^{\frac{1}{p}} \leq \|u\|_{L^p_w} < \infty ,$$

whereby $u \in L^p_w(\mathrm{d}\mu)$ and the first inequality in (3.4) holds.

Now assume that $u \in L^p_w(\mathrm{d}\mu)$, whereby $[u]_{L^p_w} < \infty$. We see from (3.2) that

$$\mu(E_u(\alpha)) \leq \frac{[u]_{L^p_w}^p}{\alpha^p} \quad \text{for every } \alpha > 0 . \tag{3.5}$$

The key new tool we will use is the so-called layer-cake decomposition of $|u(x)|$,

$$|u(x)| = \int_0^{|u(x)|} \mathrm{d}\alpha = \int_0^\infty \mathbf{1}_{\{|u(x)|>\alpha\}} \, \mathrm{d}\alpha .$$

Let $E \in \Sigma_\mu$ such that $\mu(E) \in (0,\infty)$. The Fubini-Tonelli theorem then yields

$$\int_E |u(x)| \, \mathrm{d}\mu(x) = \int_E \int_0^\infty \mathbf{1}_{\{|u(x)|>\alpha\}} \, \mathrm{d}\alpha \, \mathrm{d}\mu(x) = \int_0^\infty \int_E \mathbf{1}_{E_u(\alpha)} \, \mathrm{d}\mu(x) \, \mathrm{d}\alpha . \tag{3.6}$$

We see from (3.5) that the above inner integral can be bounded as

$$\int_E \mathbf{1}_{E_u(\alpha)} \, \mathrm{d}\mu(x) \leq \min\big\{ \mu(E) \, , \, \mu(E_u(\alpha)) \big\} \leq \min\left\{ \mu(E) \, , \, \frac{[u]_{L^p_w}^p}{\alpha^p} \right\} .$$

When this bound is placed into (3.6), and the variable of integration is rescaled appropriately, we obtain

$$\begin{aligned}
\int_E |u(x)| \, \mathrm{d}\mu(x) &\leq \int_0^\infty \int_E \mathbf{1}_{E_u(\alpha)} \, \mathrm{d}\mu(x) \, \mathrm{d}\alpha \leq \int_0^\infty \min\left\{ \mu(E) \, , \, \frac{[u]_{L^p_w}^p}{\alpha^p} \right\} \mathrm{d}\alpha \\
&= \mu(E)^{\frac{1}{p^*}} \, [u]_{L^p_w} \int_0^\infty \min\left\{ 1 \, , \, \frac{1}{\alpha^p} \right\} \mathrm{d}\alpha = \mu(E)^{\frac{1}{p^*}} \, [u]_{L^p_w} \left(\int_0^1 \mathrm{d}\alpha + \int_1^\infty \alpha^{-p} \, \mathrm{d}\alpha \right) \\
&= \mu(E)^{\frac{1}{p^*}} \, [u]_{L^p_w} \left(1 + \frac{1}{p-1} \right) = p^* \, \mu(E)^{\frac{1}{p^*}} \, [u]_{L^p_w} .
\end{aligned}$$

It then follows from definition (3.3) that

$$\|u\|_{L^p_w} = \sup_{E \in \Sigma_\mu} \left\{ \frac{1}{\mu(E)^{\frac{1}{p^*}}} \int_E |u(x)| \, \mathrm{d}\mu(x) \; : \; \mu(E) \in (0,\infty) \right\} \leq p^* \, [u]_{L^p_w} < \infty ,$$

whereby $\|u\|_{L^p_w} < \infty$ and the second inequality in (3.4) holds. $\square$

3.2. **First Hardy-Littlewood Bound.** The derivation of this bound is straightforward.

Theorem 3.2. *Let $r \in (1,\infty)$. For every kernel k that satisfies*

$$\|k\|_{L^\infty(\mathrm{d}\mu;L^r_w(\mathrm{d}\nu))} = \operatorname*{ess\,sup}_{x\in X}\left\{\sup_{E\in\Sigma_\nu}\left\{\frac{1}{\nu(E)^{\frac{1}{r^*}}}\int_E |k(x,y)|\,\mathrm{d}\nu(y)\ :\ \nu(E)\in(0,\infty)\right\}\right\} < \infty\,, \tag{3.7}$$

the integral operator $\mathcal{K}$ defined by (1.1) satisfies the bound

$$\|\mathcal{K}u\|_{L^r_w(\mathrm{d}\nu)} \le \|k\|_{L^\infty(\mathrm{d}\mu;L^r_w(\mathrm{d}\nu))}\,\|u\|_{L^1(\mathrm{d}\mu)} \quad \text{for every } u\in L^1(\mathrm{d}\mu)\,. \tag{3.8}$$

Remark. Bound (3.8) shows that the operator $\mathcal{K}$ is bounded from $L^1(\mathrm{d}\mu)$ into $L^r_w(\mathrm{d}\nu)$ with

$$\|\mathcal{K}\|_{B(L^1,L^r_w)} \le \|k\|_{L^\infty(L^r_w)} \quad \text{for every } k\in L^\infty(\mathrm{d}\mu;L^r_w(\mathrm{d}\nu))\,. \tag{3.9}$$

This result should be compared to (2.24) with $p=1$ and $q=r^*$.

Proof. For every $E\in\Sigma_\nu$ such that $\nu(E)\in(0,\infty)$ we see by Fubini-Tonelli and (3.7) that

$$\begin{aligned}
\frac{1}{\nu(E)^{\frac{1}{r^*}}}\int_E |\mathcal{K}u(y)|\,\mathrm{d}\nu(y) &\le \frac{1}{\nu(E)^{\frac{1}{r^*}}}\int_E\int |k(x,y)|\,|u(x)|\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y)\\
&= \int\left[\frac{1}{\nu(E)^{\frac{1}{r^*}}}\int_E |k(x,y)|\,\mathrm{d}\nu(y)\right]|u(x)|\,\mathrm{d}\mu(x)\\
&\le \|k\|_{L^\infty(\mathrm{d}\mu;L^r_w(\mathrm{d}\nu))}\,\|u\|_{L^1(\mathrm{d}\mu)}\,.
\end{aligned}$$

By taking the supermum over all such E and using (3.3) we obtain (3.8). □

3.3. **Second Hardy-Littlewood Bound.** The derivation of this bound again uses layer-cake decompositions, which were introduced in the proof of Theorem 3.1.

Theorem 3.3. *Let $p,q,r\in(1,\infty)$ satisfy the relation*

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 2\,. \tag{3.10}$$

There exists a positive constant $C^{p,q,r}_w$ such that for every kernel k that satisfies

$$\begin{aligned}
C_\mu &= \operatorname*{ess\,sup}_{y\in Y}\left\{\sup_{\gamma>0}\{\gamma^r\mu(E_k(\gamma))(y)\}\right\} < \infty\,,\\
C_\nu &= \operatorname*{ess\,sup}_{x\in X}\left\{\sup_{\gamma>0}\{\gamma^r\nu(E_k(\gamma))(x)\}\right\} < \infty\,,
\end{aligned} \tag{3.11}$$

where $E_k(\gamma)=\{(x,y)\in X\times Y\ :\ |k(x,y)|>\gamma\}$, the integral operator $\mathcal{K}$ defined by (1.1) satisfies the bound

$$\|\mathcal{K}u\|_{L^{q^*}_w} \le C^{p,q,r}_w\,C_\mu^{\frac{1}{p^*}}\,C_\nu^{\frac{1}{q^*}}\,[u]_{L^p_w} \quad \text{for every } u\in L^p_w(\mathrm{d}\mu)\,. \tag{3.12}$$

Here we will establish (3.12) with

$$C^{p,q,r}_w = \frac{p^*q^*r^*}{p\ r} = p^*r^*+q^*\,. \tag{3.13}$$

Remark. Notice that $C_w^{p,q,r}$ given by (3.13) is universal in the sense that it is independent of the underlying measure spaces $(X, \Sigma_\mu, \mathrm{d}\mu)$ and $(Y, \Sigma_\nu, \mathrm{d}\nu)$.

Remark. Conditions (3.11) on k are equivalent to the norm conditions

$$\|k\|_{L^\infty(\mathrm{d}\nu;L^r_w(\mathrm{d}\mu))} = \operatorname*{ess\,sup}_{y\in Y}\left\{\sup_{E\in\Sigma_\mu}\left\{\frac{1}{\mu(E)^{\frac{1}{r^*}}}\int_E |k(x,y)|\,\mathrm{d}\mu(x) \,:\, \mu(E)\in(0,\infty)\right\}\right\} < \infty\,,$$
$$\|k\|_{L^\infty(\mathrm{d}\mu;L^r_w(\mathrm{d}\nu))} = \operatorname*{ess\,sup}_{x\in X}\left\{\sup_{E\in\Sigma_\nu}\left\{\frac{1}{\nu(E)^{\frac{1}{r^*}}}\int_E |k(x,y)|\,\mathrm{d}\nu(y) \,:\, \nu(E)\in(0,\infty)\right\}\right\} < \infty\,.$$

Indeed, by following the argument that led to (3.4) we can show that

$$C_\mu^{\frac{1}{r}} \le \|k\|_{L^\infty(\mathrm{d}\nu;L^r_w(\mathrm{d}\mu))} \le r^* C_\mu^{\frac{1}{r}}\,, \qquad C_\nu^{\frac{1}{r}} \le \|k\|_{L^\infty(\mathrm{d}\mu;L^r_w(\mathrm{d}\nu))} \le r^* C_\nu^{\frac{1}{r}}\,. \tag{3.14}$$

By replacing C_μ and C_ν in the second Hardy-Littlewood bound (3.12) accordingly, we obtain

$$\|\mathcal{K}u\|_{L^{q^*}_w} \le C_w^{p,q,r}\,\|k\|^{\frac{r}{p^*}}_{L^\infty(\mathrm{d}\nu;L^r_w(\mathrm{d}\mu))}\,\|k\|^{\frac{r}{q^*}}_{L^\infty(\mathrm{d}\mu;L^r_w(\mathrm{d}\nu))}\,[u]_{L^p_w}$$
$$\text{for every } k\in L^\infty(L^r_w)(\mathrm{d}\mu,\mathrm{d}\nu) \text{ and } u\in L^p_w(\mathrm{d}\mu)\,,$$

where $C_w^{p,q,r}$ is given by (3.13). This is a weak Lebesque space analog of bound (2.30) that we obtained from the Young integral operator bound (2.29).

Proof. Because the bound (3.12) clearly holds when either $u = 0$ or $k = 0$, we only need to consider the case when $u \ne 0$ and $k \ne 0$. We can thereby normalize u so that $[u]_{L^p_w} = 1$ and assume that C_μ and C_ν are strictly positive.

Let $E \in \Sigma_\nu$ such that $\nu(E) \in (0,\infty)$. Define

$$I_E(k,u) = \int_E |\mathcal{K}u(y)|\,\mathrm{d}\nu(y)\,.$$

By the layer-cake decompositions

$$|k(x,y)| = \int_0^\infty \mathbf{1}_{\{|k(x,y)|>\gamma\}}\,\mathrm{d}\gamma\,, \qquad |u(x)| = \int_0^\infty \mathbf{1}_{\{|u(x)|>\alpha\}}\,\mathrm{d}\alpha\,,$$

and the Fubini-Tonelli Theorem we have

$$\begin{aligned} I_E(k,u) &\le \int_E\int |k(x,y)\,u(x)|\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y) \\ &\le \int_0^\infty\int_0^\infty \iint \mathbf{1}_{\{|k(x,y)|>\gamma\}}\,\mathbf{1}_{\{|u(x)|>\alpha\}}\,\mathbf{1}_E(y)\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y)\;\mathrm{d}\gamma\,\mathrm{d}\alpha\,. \end{aligned} \tag{3.15}$$

We can obtain three upper bounds of the double integral over $X\times Y$ in (3.15) by successively replacing each of the three indicator functions by 1. This procedure yields

$$\iint \mathbf{1}_{\{|k(x,y)|>\gamma\}}\,\mathbf{1}_{\{|u(x)|>\alpha\}}\,\mathbf{1}_E(y)\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y) \le U(\alpha)\,\nu(E)\,,$$
$$\iint \mathbf{1}_{\{|k(x,y)|>\gamma\}}\,\mathbf{1}_{\{|u(x)|>\alpha\}}\,\mathbf{1}_E(y)\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y) \le K_\mu(\gamma)\,\nu(E)\,,$$
$$\iint \mathbf{1}_{\{|k(x,y)|>\gamma\}}\,\mathbf{1}_{\{|u(x)|>\alpha\}}\,\mathbf{1}_E(y)\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y) \le K_\nu(\gamma)\,U(\alpha)\,,$$

where

$$U(\alpha) = \int \mathbf{1}_{\{|u(x)|>\alpha\}}\,\mathrm{d}\mu(x)\,, \qquad K_\mu(\gamma) = \operatorname*{ess\,sup}_{y\in Y}\left\{\int \mathbf{1}_{\{|k(x,y)|>\gamma\}}\,\mathrm{d}\mu(x)\right\}\,,$$
$$K_\nu(\gamma) = \operatorname*{ess\,sup}_{x\in X}\left\{\int \mathbf{1}_{\{|k(x,y)|>\gamma\}}\,\mathrm{d}\nu(y)\right\}\,. \tag{3.16}$$

By then using the minimum of these three upper bounds in (3.15) we obtain

$$I_E(k,u) \leq \int_0^\infty\!\!\int_0^\infty \min\big\{U(\alpha)\,\nu(E)\,,\, K_\mu(\gamma)\,\nu(E)\,,\, K_\nu(\gamma)\,U(\alpha)\big\}\,\mathrm{d}\gamma\,\mathrm{d}\alpha\,. \tag{3.17}$$

The normalization $[u]_{L^p_w} = 1$ and hypothesis (3.11) on k implies that for every $\alpha, \gamma \in (0,\infty)$ we have

$$U(\alpha) \leq \frac{1}{\alpha^p}\,, \qquad K_\mu(\gamma) \leq \frac{C_\mu}{\gamma^r}\,, \qquad K_\nu(\gamma) \leq \frac{C_\nu}{\gamma^r}\,.$$

When these bounds are placed into (3.17) and the variables of integration are appropriately rescaled we obtain

$$\begin{aligned} I_E(k,u) &\leq \int_0^\infty\!\!\int_0^\infty \min\left\{\frac{\nu(E)}{\alpha^p}\,,\,\frac{\nu(E)\,C_\mu}{\gamma^r}\,,\,\frac{C_\nu}{\alpha^p\gamma^r}\right\}\,\mathrm{d}\gamma\,\mathrm{d}\alpha \\ &= \nu(E)^{\frac{1}{q}}\,C_\mu^{\frac{1}{p^*}}\,C_\nu^{\frac{1}{q^*}}\;\int_0^\infty\!\!\int_0^\infty \min\left\{\frac{1}{\alpha^p}\,,\,\frac{1}{\gamma^r}\,,\,\frac{1}{\alpha^p\gamma^r}\right\}\,\mathrm{d}\gamma\,\mathrm{d}\alpha\,. \end{aligned}$$

We then see from definition (3.3) that

$$\|\mathcal{K}u\|_{L^{q^*}_w(\mathrm{d}\nu)} \leq \sup_{E\in\Sigma_\nu}\left\{\frac{1}{\nu(E)^{\frac{1}{q}}}\,I_E(k,u)\,:\,\nu(E)\in(0,\infty)\right\} \leq C_w^{p,q,r}\,C_\mu^{\frac{1}{p^*}}\,C_\nu^{\frac{1}{q^*}}\,,$$

with $C_w^{p,q,r}$ given by

$$\begin{aligned} C_w^{p,q,r} &= \int_0^\infty\!\!\int_0^\infty \min\left\{\frac{1}{\alpha^p}\,,\,\frac{1}{\gamma^r}\,,\,\frac{1}{\alpha^p\gamma^r}\right\}\,\mathrm{d}\gamma\,\mathrm{d}\alpha \\ &= \int_0^1\!\!\int_{\alpha^{\frac{p}{r}}}^\infty \gamma^{-r}\,\mathrm{d}\gamma\,\mathrm{d}\alpha + \int_0^1\!\!\int_{\gamma^{\frac{r}{p}}}^\infty \alpha^{-p}\,\mathrm{d}\alpha\,\mathrm{d}\gamma + \int_1^\infty\!\!\int_1^\infty \alpha^{-p}\,\gamma^{-r}\,\mathrm{d}\alpha\,\mathrm{d}\gamma \\ &= \frac{1}{r-1}\int_0^1 \alpha^{-\frac{p}{r^*}}\,\mathrm{d}\alpha + \frac{1}{p-1}\int_0^1 \gamma^{-\frac{r}{p^*}}\,\mathrm{d}\gamma + \frac{1}{p-1}\,\frac{1}{r-1} \\ &= \frac{1}{r-1}\,\frac{q^*}{p} + \frac{1}{p-1}\,\frac{q^*}{r} + \frac{1}{p-1}\,\frac{1}{r-1} = \frac{p^*q^*r^*}{p\;r}\,\left(\frac{1}{p^*} + \frac{1}{r^*} + \frac{1}{q^*}\right) = \frac{p^*q^*r^*}{p\;r}\,. \end{aligned}$$

Therefore the second Hardy-Littlewood bound (3.12) holds with $C_w^{p,q,r}$ given by (3.13). □

Remark. We can derive other bounds by these methods. For example, if $k \in L^\infty(L^r)(\mathrm{d}\mu\,\mathrm{d}\nu)$ for some $r \in (1,\infty)$ then for every $u \in L^r_w(\mathrm{d}\mu)$ we can show that

$$\|\mathcal{K}u\|_{L^r_w} \leq r^*\|k\|_{L^\infty(\mathrm{d}\mu;L^1(\mathrm{d}\nu))}^{\frac{1}{r^*}}\|k\|_{L^\infty(\mathrm{d}\nu;L^1(\mathrm{d}\mu))}^{\frac{1}{r}}[u]_{L^r_w}\,.$$

3.4. **Third Hardy-Littlewood Bound.** The derivation of this bound again uses layer-cake decompositions much as they were used in the proofs of Theorems 3.1 and 3.3. However here the argument will be far more technical because we will be working with classical Lebesgue spaces rather than just weak Lebesgue spaces.

Theorem 3.4. *Let $p, q, r \in (1, \infty)$ satisfy the relation*

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2\,. \tag{3.18}$$

There exists a positive constant $C^{p,q,r}$ such that for every k that satisfies

$$\begin{aligned} C_\mu &= \operatorname*{ess\,sup}_{y\in Y}\left\{\sup_{\gamma>0}\{\gamma^r \mu(E_k(\gamma))(y)\}\right\} < \infty\,, \\ C_\nu &= \operatorname*{ess\,sup}_{x\in X}\left\{\sup_{\gamma>0}\{\gamma^r \nu(E_k(\gamma))(x)\}\right\} < \infty\,, \end{aligned} \tag{3.19}$$

where $E_k(\gamma) = \{(x,y) \in X\times Y \,:\, |k(x,y)| > \gamma\}$, one has the bound

$$\begin{gathered} \iint \left|k(x,y)\,u(x)\,\overline{v(y)}\right| \mathrm{d}\mu(x)\,\mathrm{d}\nu(y) \leq C^{p,q,r}\, C_\mu^{\frac{1}{p^*}}\, C_\nu^{\frac{1}{q^*}}\, \|u\|_{L^p}\, \|v\|_{L^q} \\ \text{for every } u \in L^p(\mathrm{d}\mu) \text{ and } v \in L^q(\mathrm{d}\nu)\,, \end{gathered} \tag{3.20}$$

Here we will establish (3.20) with

$$C^{p,q,r} = \frac{r^*}{pq}\left(\frac{p^*}{r}\right)^{\frac{1}{r}+\frac{r}{p^*r^*}} \left(\frac{q^*}{r}\right)^{\frac{1}{r}+\frac{r}{q^*r^*}} \leq \frac{p^*q^*r^*}{p\;q\;r^2}\,. \tag{3.21}$$

Remark. Notice that $C^{p,q,r}$ given by (3.21) is universal in the sense that it is independent of the underlying measure spaces $(X, \Sigma_\mu, \mathrm{d}\mu)$ and $(Y, \Sigma_\nu, \mathrm{d}\nu)$.

Remark. By replacing C_μ and C_ν in the third Hardy-Littlewood bound (3.20) with r^{th}-powers of the norms $\|k\|_{L^\infty(\mathrm{d}\nu;L^r_w(\mathrm{d}\mu))}$ and $\|k\|_{L^\infty(\mathrm{d}\mu;L^r_w(\mathrm{d}\nu))}$ in accord with the bounds (3.14), we obtain

$$\begin{gathered} \iint \left|k(x,y)\,u(x)\,\overline{v(y)}\right| \mathrm{d}\mu(x)\,\mathrm{d}\nu(y) \leq C^{p,q,r}\, \|k\|_{L^\infty(\mathrm{d}\nu;L^r_w(\mathrm{d}\mu))}^{\frac{r}{p^*}}\, \|k\|_{L^\infty(\mathrm{d}\mu;L^r_w(\mathrm{d}\nu))}^{\frac{r}{q^*}}\, \|u\|_{L^p}\, \|v\|_{L^q} \\ \text{for every } k \in L^\infty(L^r_w)(\mathrm{d}\mu, \mathrm{d}\nu),\ u \in L^p(\mathrm{d}\mu), \text{ and } v \in L^q(\mathrm{d}\nu)\,, \end{gathered}$$

where $C^{p,q,r}$ is given by (3.21).

Remark. Bound (3.20) and the above remark show that the operator $\mathcal{K}$ defined by (1.1) is bounded from $L^p(\mathrm{d}\mu)$ into $L^{q^*}(\mathrm{d}\nu)$ with

$$\begin{gathered} \|\mathcal{K}\|_{B(L^p,L^{q^*})} \leq C^{p,q,r}\, C_\mu^{\frac{1}{p^*}}\, C_\nu^{\frac{1}{q^*}} \leq C^{p,q,r}\, \|k\|_{L^\infty(\mathrm{d}\nu;L^r_w(\mathrm{d}\mu))}^{\frac{r}{p^*}}\, \|k\|_{L^\infty(\mathrm{d}\mu;L^r_w(\mathrm{d}\nu))}^{\frac{r}{q^*}} \\ \text{for every } k \in L^\infty(L^r_w)(\mathrm{d}\mu, \mathrm{d}\nu)\,. \end{gathered} \tag{3.22}$$

This should be compared with bound (2.30) that we obtained from the Young integral operator bound (2.29). For each $r \in (1, \infty)$ that bound requires the kernel k to be in the more restrictive class $L^\infty(L^r)(\mathrm{d}\mu, \mathrm{d}\nu)$, but includes the cases $p = 1$ or $q = 1$. From (3.18) and (3.21) we see that $C^{p,q,r} \to \infty$ as either $(p, q^*) \to (1, r)$ or $(p, q^*) \to (r^*, \infty)$, whereby bound (3.22) breaks down in these limits. The breakdown at $(1, r)$ should be contrasted with bound (3.9), in which the range of $\mathcal{K}$ is $L^r_w(\mathrm{d}\nu)$ rather than $L^r(\mathrm{d}\nu)$.

Proof. Because the bound (3.20) clearly holds when either $u = 0$, $v = 0$, or $k = 0$, we only need to consider the case when $u \neq 0$, $v \neq 0$, and $k \neq 0$. We can then normalize u and v so that

$$\|u\|_{L^p} = \left(\int |u(x)|^p \mathrm{d}\mu(x) \right)^{\frac{1}{p}} = 1 \,, \qquad \|v\|_{L^q} = \left(\int |v(y)|^q \mathrm{d}\nu(y) \right)^{\frac{1}{q}} = 1 \,, \tag{3.23}$$

and assume that C_μ and C_ν are strictly positive.

Define

$$I(k,u,v) = \iint \left| k(x,y)\, u(x)\, \overline{v(y)} \right| \mathrm{d}\mu(x)\, \mathrm{d}\nu(y) \,.$$

For any set E, let $\mathbf{1}_E$ denote its indicator function. By the layer-cake decompositions

$$|k(x,y)| = \int_0^\infty \mathbf{1}_{\{|k(x,y)|>\gamma\}} \, \mathrm{d}\gamma \,, \quad |u(x)| = \int_0^\infty \mathbf{1}_{\{|u(x)|>\alpha\}} \, \mathrm{d}\alpha \,, \quad |v(y)| = \int_0^\infty \mathbf{1}_{\{|v(y)|>\beta\}} \, \mathrm{d}\beta \,,$$

and the Fubini-Tonelli Theorem we have

$$I(k,u,v) = \int_0^\infty \int_0^\infty \int_0^\infty \iint \mathbf{1}_{\{|k(x,y)|>\gamma\}} \, \mathbf{1}_{\{|u(x)|>\alpha\}} \, \mathbf{1}_{\{|v(y)|>\beta\}} \, \mathrm{d}\mu(x)\, \mathrm{d}\nu(y) \; \mathrm{d}\gamma \, \mathrm{d}\alpha \, \mathrm{d}\beta \,. \tag{3.24}$$

We can obtain three upper bounds of the double integral over $X \times Y$ in (3.24) by successively replacing each of the three indicator functions by 1. This procedure yields

$$\begin{aligned}
&\iint \mathbf{1}_{\{|k(x,y)|>\gamma\}} \, \mathbf{1}_{\{|u(x)|>\alpha\}} \, \mathbf{1}_{\{|v(y)|>\beta\}} \, \mathrm{d}\mu(x)\, \mathrm{d}\nu(y) \leq U(\alpha) V(\beta) \,, \\
&\iint \mathbf{1}_{\{|k(x,y)|>\gamma\}} \, \mathbf{1}_{\{|u(x)|>\alpha\}} \, \mathbf{1}_{\{|v(y)|>\beta\}} \, \mathrm{d}\mu(x)\, \mathrm{d}\nu(y) \leq K_\mu(\gamma) V(\beta) \,, \\
&\iint \mathbf{1}_{\{|k(x,y)|>\gamma\}} \, \mathbf{1}_{\{|u(x)|>\alpha\}} \, \mathbf{1}_{\{|v(y)|>\beta\}} \, \mathrm{d}\mu(x)\, \mathrm{d}\nu(y) \leq K_\nu(\gamma) U(\alpha) \,,
\end{aligned}$$

where

$$\begin{aligned}
U(\alpha) &= \int \mathbf{1}_{\{|u(x)|>\alpha\}} \, \mathrm{d}\mu(x) \,, \qquad K_\mu(\gamma) = \operatorname*{ess\,sup}_{y \in Y} \left\{ \int \mathbf{1}_{\{|k(x,y)|>\gamma\}} \, \mathrm{d}\mu(x) \right\} \,, \\
V(\beta) &= \int \mathbf{1}_{\{|v(y)|>\beta\}} \, \mathrm{d}\nu(y) \,, \qquad K_\nu(\gamma) = \operatorname*{ess\,sup}_{x \in X} \left\{ \int \mathbf{1}_{\{|k(x,y)|>\gamma\}} \, \mathrm{d}\nu(y) \right\} \,.
\end{aligned} \tag{3.25}$$

By then using the minimum of these three upper bounds in (3.24) we obtain

$$I(k,u,v) \leq \int_0^\infty \int_0^\infty \int_0^\infty \min\big\{ U(\alpha)V(\beta) \,,\, K_\mu(\gamma)V(\beta) \,,\, K_\nu(\gamma)U(\alpha) \big\} \, \mathrm{d}\gamma \, \mathrm{d}\alpha \, \mathrm{d}\beta \,. \tag{3.26}$$

Hypothesis (3.19) on k implies that for every $\gamma \in (0,\infty)$ we have

$$K_\mu(\gamma) \leq \frac{C_\mu}{\gamma^r} \,, \qquad\qquad K_\nu(\gamma) \leq \frac{C_\nu}{\gamma^r} \,.$$

When these bounds are placed into (3.26) we obtain

$$I(k,u,v) \leq \int_0^\infty \int_0^\infty \int_0^\infty \min\left\{ U(\alpha)V(\beta) \,,\, \frac{C_\mu V(\beta)}{\gamma^r} \,,\, \frac{C_\nu U(\alpha)}{\gamma^r} \right\} \, \mathrm{d}\gamma \, \mathrm{d}\alpha \, \mathrm{d}\beta \,. \tag{3.27}$$

The integral over γ in (3.27) can be evaluated exactly. When $U(\alpha)V(\beta) = 0$ it vanishes. When $0 < C_\nu U(\alpha) \leq C_\mu V(\beta)$ we obtain

$$\begin{aligned}\int_0^\infty \min\left\{U(\alpha)V(\beta)\,,\,\frac{C_\mu V(\beta)}{\gamma^r}\,,\,\frac{C_\nu U(\alpha)}{\gamma^r}\right\}\mathrm{d}\gamma &= U(\alpha)V(\beta)\int_0^{\left(\frac{C_\nu}{V(\beta)}\right)^{\frac{1}{r}}}\mathrm{d}\gamma + U(\alpha)\int_{\left(\frac{C_\nu}{V(\beta)}\right)^{\frac{1}{r}}}^\infty \frac{C_\nu}{\gamma^r}\,\mathrm{d}\gamma \\ &= C_\nu^{\frac{1}{r}} U(\alpha)V(\beta)^{\frac{1}{r^*}}\left(1+\frac{1}{r-1}\right) \\ &= r^* C_\nu^{\frac{1}{r}} U(\alpha)V(\beta)^{\frac{1}{r^*}}\,.\end{aligned}$$

Similarly, when $0 < C_\mu V(\beta) \leq C_\nu U(\alpha)$ we obtain

$$\int_0^\infty \min\left\{U(\alpha)V(\beta)\,,\,\frac{C_\mu V(\beta)}{\gamma^r}\,,\,\frac{C_\nu U(\alpha)}{\gamma^r}\right\}\mathrm{d}\gamma = r^* C_\mu^{\frac{1}{r}} U(\alpha)^{\frac{1}{r^*}} V(\beta)\,.$$

Because whenever $U(\alpha)V(\beta) > 0$ we have that $C_\nu^{\frac{1}{r}} U(\alpha)V(\beta)^{\frac{1}{r^*}} < C_\mu^{\frac{1}{r}} U(\alpha)^{\frac{1}{r^*}} V(\beta)$ if and only if $C_\nu U(\alpha) < C_\mu V(\beta)$, we see that in all cases

$$\int_0^\infty \min\left\{U(\alpha)V(\beta)\,,\,\frac{C_\mu V(\beta)}{\gamma^r}\,,\,\frac{C_\nu U(\alpha)}{\gamma^r}\right\}\mathrm{d}\gamma = r^*\min\left\{C_\nu^{\frac{1}{r}} U(\alpha)V(\beta)^{\frac{1}{r^*}}\,,\, C_\mu^{\frac{1}{r}} U(\alpha)^{\frac{1}{r^*}} V(\beta)\right\}\,.$$

When this evaluation is placed into (3.27), we obtain

$$I(k,u,v) \leq r^*\int_0^\infty\int_0^\infty \min\left\{C_\nu^{\frac{1}{r}} U(\alpha)V(\beta)^{\frac{1}{r^*}}\,,\, C_\mu^{\frac{1}{r}} U(\alpha)^{\frac{1}{r^*}} V(\beta)\right\}\mathrm{d}\beta\,\mathrm{d}\alpha\,. \tag{3.28}$$

At this point we would like to bound the right-hand side of (3.28) using only the fact that u and v satisfy the normalizations (3.23). Definitions (3.25) of $U(\alpha)$ and $V(\beta)$, the Fubini-Tonelli Theorem, and our normalizations (3.23) for u and v imply that

$$\begin{aligned}\int_0^\infty \alpha^{p-1}U(\alpha)\,\mathrm{d}\alpha &= \int_0^\infty \alpha^{p-1}\int \mathbf{1}_{\{|u(x)|>\alpha\}}\,\mathrm{d}\mu(x)\,\mathrm{d}\alpha = \iint_0^\infty \alpha^{p-1}\,\mathbf{1}_{\{|u(x)|>\alpha\}}\,\mathrm{d}\alpha\,\mathrm{d}\mu(x) \\ &= \iint_0^{|u(x)|}\alpha^{p-1}\,\mathrm{d}\alpha\,\mathrm{d}\mu(x) = \frac{1}{p}\int |u(x)|^p\,\mathrm{d}\mu(x) = \frac{1}{p}\,, \\ \int_0^\infty \beta^{q-1}V(\beta)\,\mathrm{d}\beta &= \int_0^\infty \beta^{q-1}\int \mathbf{1}_{\{|v(y)|>\beta\}}\,\mathrm{d}\nu(y)\,\mathrm{d}\beta = \iint_0^\infty \beta^{q-1}\,\mathbf{1}_{\{|v(y)|>\beta\}}\,\mathrm{d}\beta\,\mathrm{d}\nu(y) \\ &= \iint_0^{|v(y)|}\beta^{q-1}\,\mathrm{d}\beta\,\mathrm{d}\mu(y) = \frac{1}{q}\int |v(y)|^p\,\mathrm{d}\nu(y) = \frac{1}{q}\,.\end{aligned} \tag{3.29}$$

We therefore we would like to bound the right-hand side of (3.28) using only these identities.

It is clear from (3.28) that for every $\lambda \in (0,\infty)$ we have the bound

$$\begin{aligned}I(k,u,v) &\leq r^*\left[\int_0^\infty\int_0^{\lambda^{\frac{r}{q}}\alpha^{\frac{p}{q}}} C_\nu^{\frac{1}{r}} U(\alpha)V(\beta)^{\frac{1}{r^*}}\,\mathrm{d}\beta\,\mathrm{d}\alpha + \int_0^\infty\int_{\lambda^{\frac{r}{q}}\alpha^{\frac{p}{q}}}^\infty C_\mu^{\frac{1}{r}} U(\alpha)^{\frac{1}{r^*}} V(\beta)\,\mathrm{d}\beta\,\mathrm{d}\alpha\right] \\ &= r^*\left[\int_0^\infty\int_0^{\lambda^{\frac{r}{q}}\alpha^{\frac{p}{q}}} C_\nu^{\frac{1}{r}} U(\alpha)V(\beta)^{\frac{1}{r^*}}\,\mathrm{d}\beta\,\mathrm{d}\alpha + \int_0^\infty\int_0^{\lambda^{-\frac{r}{p}}\beta^{\frac{q}{p}}} C_\mu^{\frac{1}{r}} U(\alpha)^{\frac{1}{r^*}} V(\beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta\right].\end{aligned} \tag{3.30}$$

Next we apply the Hölder inequality to bound the inner integrals above as

$$
\begin{aligned}
\int_0^{\lambda^{\frac{r}{q}}\alpha^{\frac{p}{q}}} V(\beta)^{\frac{1}{r^*}}\,\mathrm{d}\beta &\le \left(\int_0^{\lambda^{\frac{r}{q}}\alpha^{\frac{p}{q}}} \beta^{-r\frac{q-1}{r^*}}\,\mathrm{d}\beta\right)^{\frac{1}{r}} \left(\int_0^\infty \beta^{q-1}V(\beta)\,\mathrm{d}\beta\right)^{\frac{1}{r^*}}\,, \\
\int_0^{\lambda^{-\frac{r}{p}}\beta^{\frac{q}{p}}} U(\alpha)^{\frac{1}{r^*}}\,\mathrm{d}\alpha &\le \left(\int_0^{\lambda^{-\frac{r}{p}}\beta^{\frac{q}{p}}} \alpha^{-r\frac{p-1}{r^*}}\,\mathrm{d}\alpha\right)^{\frac{1}{r}} \left(\int_0^\infty \alpha^{p-1}U(\alpha)\,\mathrm{d}\alpha\right)^{\frac{1}{r^*}}\,.
\end{aligned}
\tag{3.31}
$$

Because relation (3.18) implies that $r\frac{q-1}{r^*} = 1 - \frac{qr}{p^*}$ and $r\frac{p-1}{r^*} = 1 - \frac{pr}{q^*}$, we see that

$$
\begin{aligned}
\int_0^{\lambda^{\frac{r}{q}}\alpha^{\frac{p}{q}}} \beta^{-r\frac{q-1}{r^*}}\,\mathrm{d}\beta &= \frac{p^*}{qr}\left(\lambda^{\frac{r}{q}}\alpha^{\frac{p}{q}}\right)^{\frac{qr}{p^*}} = \frac{p^*}{qr}\left(\lambda^{\frac{r}{p^*}}\alpha^{p-1}\right)^{r}\,, \\
\int_0^{\lambda^{-\frac{r}{p}}\beta^{\frac{q}{p}}} \alpha^{-r\frac{p-1}{r^*}}\,\mathrm{d}\alpha &= \frac{q^*}{pr}\left(\lambda^{-\frac{r}{p}}\beta^{\frac{q}{p}}\right)^{\frac{pr}{q^*}} = \frac{q^*}{pr}\left(\lambda^{-\frac{r}{q^*}}\beta^{q-1}\right)^{r}\,.
\end{aligned}
\tag{3.32}
$$

Hence, by using (3.29) and (3.32) to evaluate the right-hand sides of (3.31), we obtain

$$
\begin{aligned}
\int_0^{\lambda^{\frac{r}{q}}\alpha^{\frac{p}{q}}} V(\beta)^{\frac{1}{r^*}}\,\mathrm{d}\beta &\le \left(\frac{p^*}{qr}\right)^{\frac{1}{r}} \lambda^{\frac{r}{p^*}}\alpha^{p-1}\left(\frac{1}{q}\right)^{\frac{1}{r^*}}\,, \\
\int_0^{\lambda^{-\frac{r}{p}}\beta^{\frac{q}{p}}} U(\alpha)^{\frac{1}{r^*}}\,\mathrm{d}\alpha &\le \left(\frac{q^*}{pr}\right)^{\frac{1}{r}} \lambda^{-\frac{r}{q^*}}\beta^{q-1}\left(\frac{1}{p}\right)^{\frac{1}{r^*}}\,.
\end{aligned}
\tag{3.33}
$$

Upon placing these results into (3.30) and again using (3.29) to evaluate the remaining integrals, we see that for every $\lambda \in (0,\infty)$ we have the bound

$$
I(k,u,v) \le \frac{r^*}{pq}\left[\left(C_\nu\,\frac{p^*}{r}\right)^{\frac{1}{r}}\lambda^{\frac{r}{p^*}} + \left(C_\mu\,\frac{q^*}{r}\right)^{\frac{1}{r}}\lambda^{-\frac{r}{q^*}}\right]. \tag{3.34}
$$

The right-hand side of (3.34) attains its minimum over $\lambda \in (0,\infty)$ when

$$
C_\nu^{\frac{1}{r}}\left(\frac{r}{p^*}\right)^{\frac{1}{r^*}}\lambda^{\frac{r}{p^*}-1} - C_\mu^{\frac{1}{r}}\left(\frac{r}{q^*}\right)^{\frac{1}{r^*}}\lambda^{-\frac{r}{q^*}-1} = 0\,.
$$

We can use the fact that $\frac{1}{p^*} + \frac{1}{q^*} = \frac{1}{r}$, which follows from relation (3.18), to solve the above equation and find that this minimum is attained at $\lambda = (C_\mu/C_\nu)^{\frac{1}{r}}(p^*/q^*)^{\frac{1}{r^*}}$. By placing this value of λ into (3.34) and again using relation (3.18), we obtain

$$
\begin{aligned}
I(k,u,v) &\le \frac{r^*}{pq}\left[\left(\frac{p^*}{r}\right)^{\frac{1}{r}}\left(\frac{p^*}{q^*}\right)^{\frac{r}{p^*r^*}} + \left(\frac{q^*}{r}\right)^{\frac{1}{r}}\left(\frac{q^*}{p^*}\right)^{\frac{r}{q^*r^*}}\right] C_\mu^{\frac{1}{p^*}}\,C_\nu^{\frac{1}{q^*}} \\
&= \frac{r^*}{pq}\left[\left(\frac{p^*}{r}\right)^{\frac{1}{r}+\frac{r}{p^*r^*}}\left(\frac{q^*}{r}\right)^{-\frac{r}{p^*r^*}} + \left(\frac{p^*}{r}\right)^{-\frac{r}{q^*r^*}}\left(\frac{q^*}{r}\right)^{\frac{1}{r}+\frac{r}{q^*r^*}}\right] C_\mu^{\frac{1}{p^*}}\,C_\nu^{\frac{1}{q^*}} \\
&= \frac{r^*}{pq}\left(\frac{p^*}{r}\right)^{\frac{1}{r}+\frac{r}{p^*r^*}}\left(\frac{q^*}{r}\right)^{\frac{1}{r}+\frac{r}{q^*r^*}}\left[\frac{r}{q^*} + \frac{r}{p^*}\right] C_\mu^{\frac{1}{p^*}}\,C_\nu^{\frac{1}{q^*}} \\
&= \frac{r^*}{pq}\left(\frac{p^*}{r}\right)^{\frac{1}{r}+\frac{r}{p^*r^*}}\left(\frac{q^*}{r}\right)^{\frac{1}{r}+\frac{r}{q^*r^*}} C_\mu^{\frac{1}{p^*}}\,C_\nu^{\frac{1}{q^*}}\,.
\end{aligned}
$$

Therefore the third Hardy-Littlewood bound (3.20) holds with $C^{p,q,r}$ given by (3.21). ☐

4. Convolution Operators

Let $(G,+)$ be an Abelian group with Haar measure $\mathrm{d}m$ defined over the σ-algebra Σ_m. (Recall that the Haar measure is a positive measure that is translation invariant; it is unique up to a positive constant factor.) Given two functions w and u defined over G, we define their *convolution* to be the function $w * u$ that is formally given by

$$w * u(y) = \int w(y-x)\, u(x)\, \mathrm{d}m(x)\,. \tag{4.1}$$

This can be viewed as an integral operator of the form (1.1) where $X = Y = G$, $\mathrm{d}\mu = \mathrm{d}\nu = \mathrm{d}m$, $\Sigma_\mu = \Sigma_\nu = \Sigma_m$ and $k(x,y) = w(y-x)$. Such operators are called *convolution operators*. In this setting, w is called the *convolution kernel*. In this section we derive bounds that ensure the convolution (4.1) maps between either classical Lebesgue spaces or weak Lebesgue spaces. We will start by specializing the Young integral operator bound (2.29), and the Hardy-Littlewood bounds (3.8), (3.12) and (3.20) to this setting. We will then give a Calderon-Zygmund bound in the setting where $G = \mathbb{R}^D$ and $\mathrm{d}m$ is Lebesque measure over $\mathbb{R}^D$.

4.1. **Young Convolution Inequality.** The Young convolution inequality follows directly from Young integral operator bound (2.29).

Corollary 4.1. *Let $p, q, r \in [1,\infty]$ satisfy the relation*

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2\,. \tag{4.2}$$

For every $u \in L^p(\mathrm{d}m)$, $v \in L^q(\mathrm{d}m)$, and $w \in L^r(\mathrm{d}m)$ we have

$$\iint \big| w(y-x)\, u(x)\, \overline{v(y)} \big|\, \mathrm{d}m(x)\, \mathrm{d}m(y) \leq \|u\|_{L^p}\, \|v\|_{L^q}\, \|w\|_{L^r}\,. \tag{4.3}$$

Proof. Because $k(x,y) = w(y-x)$ and $w \in L^r(\mathrm{d}m)$, we see that $k \in L^\infty(L^r)(\mathrm{d}m, \mathrm{d}m)$ with $\|k\|_{L^\infty(\mathrm{d}m; L^r(\mathrm{d}m))} = \|w\|_{L^r(\mathrm{d}m)} < \infty$. Because relation (4.2) implies both that $p, q \in [1, r^*]$ and that relation (2.28) holds, it therefore follows from (2.29) that the Young convolution inequality (4.3) holds. $\square$

4.2. **Hardy-Littlewood-Sobolev Inequalities.** These inequalities are specializations of the Hardy-Littlewood inequalities to convolution kernels $k(x,y) = w(y-x)$. In the Young convolution inequality (4.3) the function w sits in $L^r(\mathrm{d}m)$. When $r \in (1,\infty)$ the Hardy-Littlewood-Sobolev inequalities allows this class to be extended to $L^r_w(\mathrm{d}m)$.

The first Hardy-Littlewood-Sobolev inequality is an immediate corollary of the first Hardy-Littlewood bound (3.8).

Corollary 4.2. *Let $r \in (1,\infty)$. For every $u \in L^1(\mathrm{d}m)$ and $w \in L^r_w(\mathrm{d}m)$ we have*

$$\|w * u\|_{L^r_w} \leq \|u\|_{L^1}\, \|w\|_{L^r_w}\,. \tag{4.4}$$

Proof. Because $k(x,y) = w(y-x)$ and $w \in L^r_w(\mathrm{d}m)$, we see that $k \in L^\infty(\mathrm{d}m; L^r_w(\mathrm{d}m))$ with $\|k\|_{L^\infty(\mathrm{d}m; L^r_w(\mathrm{d}m))} = \|w\|_{L^r_w(\mathrm{d}m)} < \infty$. It therefore follows from (3.8) that the first Hardy-Littlewood-Sobolev inequality (4.4) holds. $\square$

The second Hardy-Littlewood-Sobolev inequality is an immediate corollary of the second Hardy-Littlewood bound (3.12).

Corollary 4.3. *Let $p, q, r \in (1, \infty)$ that satisfy relation (4.2). Then there exists a positive constant $C^{p,q,r}_{G,w}$ such that for every $u \in L^p_w(\mathrm{d}m)$ and $w \in L^r_w(\mathrm{d}m)$ we have*

$$\|w * u\|_{L^{q^*}_w} \leq C^{p,q,r}_{G,w}\, [u]_{L^p_w}\, [w]_{L^r_w} \,. \tag{4.5}$$

Here we will establish (4.5) with

$$C^{p,q,r}_{G,w} = \frac{p^* q^* r^*}{p\ q} \,. \tag{4.6}$$

Remark. Notice that $C^{p,q,r}_{G,w}$ given by (4.6) is universal in the sense that it is independent of G.

Proof. Because $k(x, y) = w(y - x)$ and $w \in L^r_w(\mathrm{d}m)$, we see that $k \in L^\infty(\mathrm{d}m; L^r_w(\mathrm{d}m))$ with $C_m = [w]_{L^r_w(\mathrm{d}m)} < \infty$. Because relations (4.2) and (3.10) are the same, it follows from (3.12) that the second Hardy-Littlewood-Sobolev inequality (4.5) holds. Formula (4.6) for $C^{p,q,r}_{G,w}$ follows from formula (3.13). □

The third Hardy-Littlewood-Sobolev inequality is an immediate corollary of the third Hardy-Littlewood bound (3.20).

Corollary 4.4. *Let $p, q, r \in (1, \infty)$ that satisfy relation (4.2). Then there exists a positive constant $C^{p,q,r}_{G}$ such that for every $u \in L^p(\mathrm{d}m)$, $v \in L^q(\mathrm{d}m)$, and $w \in L^r_w(\mathrm{d}m)$ we have*

$$\iint \left| w(y - x)\, u(x)\, \overline{v(y)} \right| \mathrm{d}m(x)\, \mathrm{d}m(y) \leq C^{p,q,r}_{G}\, \|u\|_{L^p}\, \|v\|_{L^q}\, [w]_{L^r_w} \,. \tag{4.7}$$

Here we will establish (4.7) with

$$C^{p,q,r}_{G} = \frac{r^*}{pq} \left(\frac{p^*}{r}\right)^{\frac{1}{r} + \frac{r}{p^* r^*}} \left(\frac{q^*}{r}\right)^{\frac{1}{r} + \frac{r}{q^* r^*}} \leq \frac{p^* q^* r^*}{p\ q\ r^2} \,. \tag{4.8}$$

Remark. Notice that $C^{p,q,r}_{G}$ given by (4.8) is universal in the sense that it is independent of G. For a discussion of sharp values for $C^{p,q,r}_{G}$ when $G = \mathbb{R}^D$ see [3].

Proof. Because $k(x, y) = w(y - x)$ and $w \in L^r_w(\mathrm{d}m)$, we see that $k \in L^\infty(\mathrm{d}m; L^r_w(\mathrm{d}m))$ with $C_m = [w]_{L^r_w(\mathrm{d}m)} < \infty$. Because relations (4.2) and (3.18) are the same, it follows from (3.20) that the third Hardy-Littlewood-Sobolev inequality (4.7) holds. Formula (4.8) for $C^{p,q,r}_{G}$ follows from formula (3.21). □

4.3. **Calderon-Zygmund Inequality.** The preceding theory cannot be applied to integral operators with more singular kernels such as the classical Hilbert transform $\mathcal{H}$, which is fomally defined for functions over $\mathbb{R}$ by

$$\mathcal{H}u(y) = \frac{\mathrm{PV}}{\pi} \int_{-\infty}^{\infty} \frac{u(y - x)}{x} \, \mathrm{d}x \,. \tag{4.9}$$

Here the PV indicates that the integral is understood in the sense of *principle value*, namely — as the limit

$$\mathrm{PV} \int_{-\infty}^{\infty} \frac{w(x)}{x} \, \mathrm{d}x = \lim_{\epsilon \to 0^+} \left[\int_{-\infty}^{-\epsilon} \frac{w(x)}{x} \, \mathrm{d}x + \int_{\epsilon}^{\infty} \frac{w(x)}{x} \, \mathrm{d}x \right] . \tag{4.10}$$

Such a limit will exist when w sufficiently regular near $x = 0$ because $1/x$ takes different signs on either side of $x = 0$, which leads to cancellation. Calderon-Zygmund theory bounds integral operators with singular convolution kernels that are on the borderline of being locally integrable provided there is cancellation near the singularity. We will not give a very general result here. Rather, we will give without proof a special result that has wide applicability [4].

We specialize to the case in which $G = \mathbb{R}^D$ and $\mathrm{d}m$ is the usual Lebesgue measure on $\mathbb{R}^D$. Calderon-Zygmund theory implies the following.

Theorem 4.1. *Let w be a complex-valued function over $\mathbb{R}^D$ that has the factored form*

$$w(z) = h(|z|)\, j\left(\frac{z}{|z|}\right) , \tag{4.11}$$

where h is Lipschitz continuous away from $z = 0$ and satisfies $\sup\{|z|^D |h(|z|)| : |z| > 0\} < \infty$, *while j is Lipschitz continuous over $\mathbb{S}^{D-1}$ and satisfies the cancellation condition*

$$\int_{\mathbb{S}^{D-1}} j(o)\, \mathrm{d}S(o) = 0 . \tag{4.12}$$

Here $\mathrm{d}S$ denotes the usual Lebesgue surface measure on $\mathbb{S}^{D-1}$. For every $\epsilon > 0$ define the function w_ϵ by $w_\epsilon(z) = \mathbf{1}_{\{|z|>\epsilon\}}\, w(z)$, and the operator $\mathcal{K}_\epsilon$ by

$$\mathcal{K}_\epsilon u(y) = \int w_\epsilon(y - x)\, u(x)\, \mathrm{d}m(x) . \tag{4.13}$$

Then for every $p \in (1, \infty)$ there exists a positive constant C_p that is independent of ϵ such that for every $\epsilon > 0$ the operator $\mathcal{K}_\epsilon$ satisfies the bound

$$\|\mathcal{K}_\epsilon u\|_{L^p} \leq C_p \, \|u\|_{L^p} \quad \textit{for every } u \in L^p(\mathrm{d}m) , \tag{4.14}$$

Moreover, for every $u \in L^p(\mathrm{d}m)$ the limit

$$\mathcal{K}u = \lim_{\epsilon \to 0} \mathcal{K}_\epsilon u \quad \textit{exists in } L^p(\mathrm{d}m) , \tag{4.15}$$

and the operator $\mathcal{K}$ so defined satisfies the bound

$$\|\mathcal{K}u\|_{L^p} \leq C_p \, \|u\|_{L^p} \quad \textit{for every } u \in L^p(\mathrm{d}m) . \tag{4.16}$$

4.4. **Summary.** Our results regarding the convolution of two functions are summarized in the following table.

$$\begin{aligned}
L^p * L^q &\subset L^r && \text{for } p, q, r \in [1, \infty] \text{ such that } \tfrac{1}{p} + \tfrac{1}{q} = 1 + \tfrac{1}{r} . \\
L^r_w * L^1 &\subset L^r_w && \text{for } r \in (1, \infty) . \\
L^p_w * L^q_w &\subset L^r_w && \text{for } p, q, r \in (1, \infty) \text{ such that } \tfrac{1}{p} + \tfrac{1}{q} = 1 + \tfrac{1}{r} . \\
L^p_w * L^q &\subset L^r && \text{for } p, q, r \in (1, \infty) \text{ such that } \tfrac{1}{p} + \tfrac{1}{q} = 1 + \tfrac{1}{r} . \\
CZ * L^r &\subset L^r && \text{for } r \in (1, \infty) .
\end{aligned}$$

The first item follows from the Young convolution inequality, the second from the first Hardy-Littlewood-Sobolev inequality, the third from the second Hardy-Littlewood-Sobolev inequality, the fourth from the third Hardy-Littlewood-Sobolev inequality, and the last from the Calderon-Zygmund inequality, where CZ denotes all functions of the Calderon-Zygmund form (4.11).

Remark. The range given for the second item in the table cannot be reduced to L^r so as to be consistent with the fourth item with $q = 1$. Indeed, let $u(x) = |x|^{-\frac{D}{r}}$ for some $r \in (1, \infty)$ and $v(x)$ be a positive, smooth, rapidly decreasing function. One can show that

$$\lim_{x \to \infty} |x|^{\frac{D}{r}} \int |x - y|^{-\frac{D}{r}}\, v(y)\, \mathrm{d}m(y) = \int v(y)\, \mathrm{d}m(y) > 0 .$$

Hence, $u \in L^r_w$, $v \in L^1$, but $u * v \notin L^r$.

Remark. The range given for the third item in the table cannot be reduced to L^r. Indeed, let $p, q, r \in (1,\infty)$ such that $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$. Set $u(x)=|x|^{-\frac{D}{p}}$ and $v(x)=|x|^{-\frac{D}{q}}$. One can show that for some $C>0$

$$u * v(x) = \int |x-y|^{-\frac{D}{p}}\,|y|^{-\frac{D}{q}}\,\mathrm{d}m(y) = C\,|x|^{-\frac{D}{r}}\,.$$

Hence, $u \in L^p_w$, $v \in L^q_w$, but $u * v \notin L^r$.

5. Examples from Partial Differential Equations

Here we apply the foregoing theory to examples of integral operators that arise in the study of partial differential equations.

Example. For $D>2$ the Green function g of $-\Delta_x$ over $\mathbb{R}^D$ is given by

$$g(x) = \frac{1}{|\mathbb{S}^{D-1}|}\,|x|^{-D+2}\,.$$

If u is the solution of the Poisson equation $-\Delta_x u = f$ for some $f \in L^p(\mathrm{d}m)$ then formally

$$u = g * f\,, \qquad \nabla_x u = (\nabla_x g) * f\,, \qquad \nabla_x^2 u = (\nabla_x^2 g) * f\,,$$

where

$$\nabla_x g(x) = -\frac{D-2}{|\mathbb{S}^{D-1}|}\,|x|^{-D+1}\,\frac{x}{|x|}\,, \qquad \nabla_x^2 g(x) = \frac{D-2}{|\mathbb{S}^{D-1}|}\,|x|^{-D}\left(D\frac{x\otimes x}{|x|^2} - I\right)\,.$$

Because

$$|\nabla_x g(x)| = \frac{D-2}{|\mathbb{S}^{D-1}|}\,|x|^{-D+1}\,, \qquad |\nabla_x^2 g(x)| = \frac{(D-2)(D-1)}{|\mathbb{S}^{D-1}|}\,|x|^{-D}\,,$$

we see that

$$g \in L_w^{\frac{D}{D-2}}(\mathrm{d}m)\,, \qquad \nabla_x g \in L_w^{\frac{D}{D-1}}(\mathrm{d}m)\,, \qquad \nabla_x^2 g \in CZ(\mathrm{d}m)\,,$$

where $CZ(\mathrm{d}m)$ denotes the set of all functions that have the Calderon-Zygmund form (4.11). Hence, if $f \in L^p(\mathrm{d}m)$ then

$$\begin{aligned} u &\in L^{\frac{pD}{D-2p}}(\mathrm{d}m) && \text{when } p \in (1, \tfrac{D}{2})\,,\\ \nabla_x u &\in L^{\frac{pD}{D-p}}(\mathrm{d}m) && \text{when } p \in (1, D)\,,\\ \nabla_x^2 u &\in L^p(\mathrm{d}m) && \text{when } p \in (1,\infty)\,. \end{aligned}$$

The last result shows that solutions of the Poisson equation gain two derivatives.

Example. The Green function g of $-\Delta_x+\kappa^2$ over $\mathbb{R}^3$ is given by

$$g(x) = \frac{1}{4\pi}\,\frac{e^{-\kappa|x|}}{|x|}\,.$$

If u is the solution of $-\Delta_x u + \kappa^2 u = f$ for some $f \in L^p(\mathrm{d}m)$ then formally

$$u = g * f\,, \qquad \nabla_x u = (\nabla_x g) * f\,, \qquad \nabla_x^2 u = (\nabla_x^2 g) * f\,,$$

where

$$\begin{aligned} \nabla_x g(x) &= -\frac{1}{4\pi}\,\frac{e^{-\kappa|x|}}{|x|^2}\,(1+\kappa|x|)\,\frac{x}{|x|}\,,\\ \nabla_x^2 g(x) &= \frac{1}{4\pi}\,\frac{e^{-\kappa|x|}}{|x|^3}\,(1+\kappa|x|)\left(3\frac{x\otimes x}{|x|^2} - I\right) + \frac{\kappa^2}{4\pi}\,\frac{e^{-\kappa|x|}}{|x|}\,\frac{x\otimes x}{|x|^2}\,. \end{aligned}$$

Because

$$|\nabla_x g(x)| = \frac{1}{4\pi} \frac{e^{-\kappa|x|}}{|x|^2} (1 + \kappa|x|) \,,$$

we see that

$$g \in L^q(\mathrm{d}m) \quad \text{for every } q \in [1,3) \qquad \text{and} \qquad g \in L^3_w(\mathrm{d}m)\,,$$
$$\nabla_x g \in L^q(\mathrm{d}m) \quad \text{for every } q \in [1,\tfrac{3}{2}) \qquad \text{and} \qquad \nabla_x g \in L^{\frac{3}{2}}_w(\mathrm{d}m)\,.$$

Hence, if $f \in L^p(\mathrm{d}m)$ then

$$u \in L^r(\mathrm{d}m) \quad \begin{cases} \text{for every } r \in [p,\infty] & \text{when } p \in (\frac{3}{2},\infty)\,, \\ \text{for every } r \in [p,\infty) & \text{when } p = \frac{3}{2}\,, \\ \text{for every } r \in [p,\frac{3p}{3-2p}) & \text{when } p \in (1,\frac{3}{2})\,, \\ \text{for every } r \in [1,3) & \text{when } p = 1\,, \end{cases}$$

$$\nabla_x u \in L^r(\mathrm{d}m) \quad \begin{cases} \text{for every } r \in [p,\infty] & \text{when } p \in (3,\infty)\,, \\ \text{for every } r \in [p,\infty) & \text{when } p = 3\,, \\ \text{for every } r \in [p,\frac{3p}{3-p}) & \text{when } p \in (1,3)\,, \\ \text{for every } r \in [1,\frac{3}{2}) & \text{when } p = 1\,, \end{cases}$$

In particular, we see that $u \in L^p(dm)$ and $\nabla_x u \in L^p(dm)$.

Finally, notice that $\nabla_x^2 g = H_1(x) + H_2(x)$ where H_1 and H_2 are the matrix-valued functions

$$H_1(x) = \frac{1}{4\pi} \frac{e^{-\kappa|x|}(1+\kappa|x|)}{|x|^3} \left(3\frac{x\otimes x}{|x|^2} - I \right)\,, \qquad H_2(x) = \frac{\kappa^2}{4\pi} \frac{e^{-\kappa|x|}}{|x|} \frac{x \otimes x}{|x|^2}\,.$$

Because

$$|H_1(x)| = \frac{1}{2\pi} \frac{e^{-\kappa|x|}(1+\kappa|x|)}{|x|^3}\,, \qquad |H_2(x)| = \frac{\kappa^2}{4\pi} \frac{e^{-\kappa|x|}}{|x|}\,,$$

we see that $H_1 \in CZ(\mathrm{d}m)$ while

$$H_2 \in L^q(\mathrm{d}m) \quad \text{for every } q \in [1,3)\,, \qquad \text{and} \qquad H_2 \in L^3_w(\mathrm{d}m)\,.$$

In particular, we see that if $f \in L^p(\mathrm{d}m)$ then

$$\nabla_x^2 u = H_1 * f + H_2 * f \in L^p(\mathrm{d}m)\,, \qquad \text{when } p \in (1,\infty)\,.$$

Therefore, as with the Poisson equation, solutions of $-\Delta_x u + \kappa^2 u = f$ gain two derivatives.

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