First In-Class Exam Solutions
Math 246, Spring 2009, Professor David Levermore

(1) [12] Suppose you have used a numerical method to approximate the solution of an initial-value problem over the time interval $[0, 5]$ with 1000 uniform time steps. About how many uniform time steps do you need to reduce the global error of your approximation by a factor of 81 if the method you had used was each of the following?

(a) Runge-Kutta method
Solution: This method is fourth order, so its global error scales like h^4. To reduce the error by a factor of 81, you must reduce h by a factor of $81^{\frac{1}{4}} = 3$. You must therefore increase the number of time steps by a factor of 3, which means you need 3000 uniform time steps.

(b) Heun-midpoint method
Solution: This method is second order, so its global error scales like h^2. To reduce the error by a factor of 81, you must reduce h by a factor of $81^{\frac{1}{2}} = 9$. You must therefore increase the number of time steps by a factor of 9, which means you need 9000 uniform time steps.

(c) Heun-trapezoidal method
Solution: This method is second order, so its global error scales like h^2. To reduce the error by a factor of 81, you must reduce h by a factor of $81^{\frac{1}{2}} = 9$. You must therefore increase the number of time steps by a factor of 9, which means you need 9000 uniform time steps.

(d) Euler method
Solution: This method is first order, so its global error scales like h. To reduce the error by a factor of 81, you must reduce h by a factor of 81. You must therefore increase the number of time steps by a factor of 81, which means you need 81000 uniform time steps.

(2) [20] Find the explicit solution for each of the following initial-value problems and identify its interval of existence (interval of definition).

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{e^x}{1+y}\,, \quad y(0) = -2.$

Solution: This equation is separable. Its separated differential form is

$$(y+1)\,\mathrm{d}y = e^x\,\mathrm{d}x\,, \qquad \Longrightarrow \qquad \tfrac{1}{2}(y+1)^2 = e^x + c\,.$$

The initial condition $y(0) = -2$ implies that $c = \frac{1}{2}(-2+1)^2 - e^0 = \frac{1}{2} - 1 = -\frac{1}{2}$. Therefore $(y+1)^2 = 2e^x - 1$, which can be solved as

$$z = -1 - \sqrt{2e^x - 1}\,, \quad \text{with interval of existence } x > \log(\tfrac{1}{2})\,.$$

The negative square root is needed to satisfy the initial condition.

(b) $\dfrac{\mathrm{d}u}{\mathrm{d}t} = \dfrac{t^3 - u}{1+t}\,, \quad u(2) = 3.$

Solution: This equation is linear. Its linear normal form is

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{1}{1+t}\,u = \frac{t^3}{1+t}\,.$$

An integrating factor is $\exp\left(\int_0^t \frac{1}{1+s}\,ds\right) = \exp(\log(1+t)) = 1+t$, so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big((1+t)u\big) = (1+t)\cdot\frac{t^3}{1+t} = t^3\,, \qquad \Longrightarrow \qquad (1+t)u = \tfrac{1}{4}t^4 + c\,.$$

The initial condition $u(2) = 3$ implies that $c = (1+2)\cdot 3 - \frac{1}{4}2^4 = 9 - 4 = 5$. Therefore

$$u = \frac{\frac{1}{4}t^4 + 5}{1+t}\,, \quad \text{with interval of existence } t > -1\,.$$

(3) [16] Consider the differential equation $\dfrac{\mathrm{d}x}{\mathrm{d}t} = x(2-x)(4-x)^2$.

(a) Sketch its phase-line. Indicate all of the stationary (equilibrium) solutions and classify each as being either stable, unstable, or semistable.
(b) If $x(0) = 6$, how does the solution $x(t)$ behave as $t \to \infty$?
(c) If $x(0) = 3$, how does the solution $x(t)$ behave as $t \to \infty$?
(d) If $x(0) = 1$, how does the solution $x(t)$ behave as $t \to \infty$?
(e) If $x(0) = -2$, how does the solution $x(t)$ behave as $t \to \infty$?

Solution (a): The stationary solutions are $x = 0$, $x = 2$, and $x = 4$. A sign analysis of $x(2-x)(4-x)^2$ shows that the phase-line for this equation is therefore

$$\begin{array}{ccccccc} - & & + & & - & & - \\ \leftarrow\leftarrow\leftarrow\leftarrow & \bullet & \rightarrow\rightarrow\rightarrow\rightarrow & \bullet & \leftarrow\leftarrow\leftarrow\leftarrow & \bullet & \leftarrow\leftarrow\leftarrow\leftarrow\ y \\ & 0 & & 2 & & 4 & \\ & \text{unstable} & & \text{stable} & & \text{semistable} & \end{array}$$

(b): The phase-line shows that if $x(0) = 6$ then $x(t) \to 4$ as $t \to \infty$.
(c): The phase-line shows that if $x(0) = 3$ then $x(t) \to 2$ as $t \to \infty$.
(d): The phase-line shows that if $x(0) = 1$ then $x(t) \to 2$ as $t \to \infty$.
(e): The phase-line shows that if $x(0) = -2$ then $x(t) \to -\infty$ as $t \to \infty$.

(4) [16] Consider the following MATLAB function M-file.

```
function [t,y] = solveit(ti, yi, tf, n)

h = (tf - ti)/n;
t = zeros(n + 1, 1);
y = zeros(n + 1, 1);
t(1) = ti;
y(1) = yi;
for k = 1:n
yhalf = y(k) + (h/2)*(2*y(k) - (y(k))^2);
t(k + 1) = t(k) + h;
y(k + 1) = y(k) + h*(2*yhalf - (yhalf)^2);
end
```

(a) What is the initial-value problem being approximated numerically?
(b) What is the numerical method being used?
(c) What are the output values of t(2) and y(2) that you would expect for input values of ti = 1, yi = 1, tf = 5, n = 20?

Solution (a): The initial-value problem being approximated numerically is

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2y - y^2 \,, \quad y(\mathrm{ti}) = \mathrm{yi} \,.$$

(b): It is being approximated by the Heun-midpoint method.
(c): When ti = 1, yi = 1, tf = 5, n = 20 one has h = (tf − ti)/n = (5 − 1)/20 = .2, t(1) = ti = 1, and y(1) = yi = 1.
Setting k = 1 inside the "for" loop then yields

$$\begin{aligned} &\mathrm{yhalf} = \mathrm{y}(1) + (\mathrm{h}/2)\ (2\ \mathrm{y}(1) - \mathrm{y}(1)^2) = 1 + .1\,(2 \cdot 1 - 1) = 1.1 \,, \\ &\mathrm{t}(2) = \mathrm{t}(1) + \mathrm{h} = 1 + .2 = 1.2 \,, \\ &\mathrm{y}(2) = \mathrm{y}(1) + \mathrm{h}\ (2\ \mathrm{yhalf} - \mathrm{yhalf}^2) = 1 + .2\,(2 \cdot 1.1 - (1.1)^2) \,. \end{aligned}$$

The above answer got full credit, but y(2) = 1.198 if you worked out the arithmetic.

(5) [16] A student borrows \$6000 at an interest rate of 10% per year compounded continuously. Assume that the student makes payments continuously at a constant rate of k dollars per year. Let $B(t)$ denote the balance of the loan at t years.
(a) Write down an initial-value problem that governs $B(t)$ at any positive time for which the balance is still positive.
(b) Determine the value of k required to pay off the loan in five years.

Solution (a): The balance $B(t)$ satisfies the initial-value problem

$$\frac{\mathrm{d}B}{\mathrm{d}t} = .1B - k \,, \qquad B(0) = 6000 \,.$$

(b): The equation is linear and can be put into the integrating factor form

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\big(e^{-.1t}B\big) = -ke^{-.1t} \,, \qquad &\Longrightarrow \qquad e^{-.1t}B(t) = 10ke^{-.1t} + c \,, \\ &\Longrightarrow \qquad B(t) = 10k + ce^{.1t} \,. \end{aligned}$$

The initial condition $B(0) = 6000$ implies that $c = 6000 - 10k$. Hence,

$$B(t) = 10k(1 - e^{.1t}) + 6000e^{.1t} \,.$$

Paying off the loan in five years means that $B(5) = 0$. Therefore k must satisfy

$$0 = 6000e^{.5} - 10k(e^{.5} - 1) \,, \qquad \Longrightarrow \qquad k = \frac{600e^{.5}}{e^{.5} - 1} \,.$$

(6) [20] Give an implicit general solution to each of the following differential equations.
(a) $2xy\,\mathrm{d}x + (2x^2 + e^y)\,\mathrm{d}y = 0$.

Solution: This differential form is *not exact* because

$$\partial_y(2xy) = 2x \quad \neq \quad \partial_x(2x^2 + e^y) = 4x \,.$$

You therefore seek an *integrating factor* μ such that

$$\partial_y[2xy\mu] = \partial_x[(2x^2 + e^y)\mu] \,.$$

Expanding the partial derivatives yields

$$2xy\partial_y\mu + 2x\mu = (2x^2 + e^y)\partial_x\mu + 4x\mu\,.$$

If you set $\partial_x\mu = 0$ then this becomes

$$2xy\partial_y\mu + 2x\mu = 4x\mu\,,$$

which reduces to $y\partial_y\mu = \mu$. This has the normal form

$$\partial_y\mu - \frac{1}{y}\,\mu\,, \qquad \Longrightarrow \qquad \partial_y\left(\frac{\mu}{y}\right) = 0\,,$$

which yields the integrating factor $\mu = y$.

Because y is an integrating factor, the differential form

$$2xy^2\,\mathrm{d}x + (x^2y + ye^y)\,\mathrm{d}y = 0 \quad \text{is exact}\,.$$

You can therefore find $H(x,y)$ such that

$$\partial_x H(x,y) = 2xy^2\,, \qquad \partial_y H(x,y) = x^2y + ye^y\,.$$

Integrating the first equation with respect to x yields

$$H(x,y) = \int 2xy^2\,\mathrm{d}x = x^2y^2 + h(y)\,.$$

Plugging this expression for $H(x,y)$ into the second equation gives

$$2x^2y + h'(y) = \partial_y H(x,y) = 2x^2y + ye^y\,,$$

which yields $h'(y) = ye^y$. One integration by parts then yields

$$h(y) = \int ye^y\mathrm{d}y = ye^y - \int e^y\,\mathrm{d}y = ye^y - e^y - c\,.$$

Taking $h(y) = (y-1)e^y$, a general solution is therefore given implicitly by

$$x^2y^2 + (y-1)e^y = c\,.$$

(b) $(3x^2y^2 + 5x^4)\,\mathrm{d}x + (2x^3y + 4y^3)\,\mathrm{d}y = 0\,.$

Solution: This differential form is *exact* because

$$\partial_y(3x^2y^2 + 5x^4) = 6x^2y \quad = \quad \partial_x(2x^3y + 4y^3) = 6x^2y\,.$$

We can therefore find $H(x,y)$ such that

$$\partial_x H(x,y) = 3x^2y^2 + 5x^4\,, \qquad \partial_y H(x,y) = 2x^3y + 4y^3\,.$$

Integrating the second equation with respect to y yields

$$H(x,y) = \int (2x^3y + 4y^3)\,\mathrm{d}x = x^3y^2 + y^4 + h(x)\,.$$

Plugging this expression for $H(x,y)$ into the first equation gives

$$3x^2y^2 + h'(x) = \partial_x H(x,y) = 3x^2y^2 + 5x^4\,,$$

which yields $h'(x) = 5x^4$. Taking $h(x) = x^5$, a general solution is therefore given implicitly by

$$x^3y^2 + y^4 + x^5 = c\,.$$