

**Second In-Class Exam Solutions**  
**Math 246, Spring 2009, Professor David Levermore**

- (1) [4] Give the interval of existence for the solution of the initial-value problem

$$\sin(t) \frac{d^4 z}{dt^4} + \frac{1+t^2}{1-t} \frac{dz}{dt} = \frac{e^t}{5-t}, \quad z(2) = z'(2) = z''(2) = z'''(2) = 0.$$

**Solution.** The normal form of the equation is

$$\frac{d^4 z}{dt^4} + \frac{1+t^2}{(1-t)\sin(t)} \frac{dz}{dt} = \frac{e^t}{(5-t)\sin(t)},$$

The coefficient and forcing are both continuous over the interval  $(1, \pi)$ , which contains the initial time  $t = 2$ . The coefficient is not defined at  $t = 1$  while both the coefficient and the forcing are not defined at  $t = \pi$ . The interval of existence is therefore  $(1, \pi)$ .

- (2) [8] Let  $L$  be a linear ordinary differential operator with constant coefficients. Suppose that all the roots of its characteristic polynomial (listed with their multiplicities) are  $4 + i3$ ,  $4 + i3$ ,  $4 - i3$ ,  $4 - i3$ ,  $-2$ ,  $-2$ ,  $-2$ ,  $7$ ,  $0$ ,  $0$ .

- (a) Give the order of  $L$ .

**Solution.** Because there are 10 roots listed, the degree of the characteristic polynomial is 10, whereby the order of  $L$  is 10.

- (b) Give a general real solution of the homogeneous equation  $Ly = 0$ .

**Solution.** A general solution is

$$y = c_1 e^{4t} \cos(3t) + c_2 e^{4t} \sin(3t) + c_3 t e^{4t} \cos(3t) + c_4 t e^{4t} \sin(3t) \\ + c_5 e^{-2t} + c_6 t e^{-2t} + c_7 t^2 e^{-2t} + c_8 e^{7t} + c_9 + c_{10} t.$$

The reasoning is as follows:

- the double conjugate pair  $4 \pm i3$  yields  $e^{4t} \cos(3t)$ ,  $e^{4t} \sin(3t)$ ,  $t e^{4t} \cos(3t)$ , and  $t e^{4t} \sin(3t)$ ;
- the triple real root  $-2$  yields  $e^{-2t}$ ,  $t e^{-2t}$ , and  $t^2 e^{-2t}$ ;
- the single real root  $7$  yields  $e^{7t}$ ;
- the double real root  $0$  yields  $1$  and  $t$ .

- (3) [8] Let  $D = \frac{d}{dt}$ . Consider the equation

$$Ly = D^2 y + 4Dy + 13y = \sin(t^2).$$

- (a) Compute the Green function  $g(t)$  associated with  $L$ .

**Solution.** The Green function  $g(t)$  satisfies

$$D^2 g + 4Dg + 13g = 0, \quad g(0) = 0, \quad g'(0) = 1.$$

This initial-value problem may be solved either directly or by Laplace transform.

**Directly.** The characteristic polynomial of L is  $p(z) = z^2 + 4z + 13 = (z+2)^2 + 3^2$ , which has roots  $-2 \pm i3$ . Set  $g(t) = c_1 e^{-2t} \cos(3t) + c_2 e^{-2t} \sin(3t)$ . The first initial condition implies  $g(0) = c_1 = 0$ , whereby  $g(t) = c_2 e^{-2t} \sin(3t)$ . Because

$$g'(t) = 3c_2 e^{-2t} \cos(3t) - 2c_2 e^{-2t} \sin(3t),$$

the second initial condition implies  $g'(0) = 3c_2 = 1$ , whereby  $c_2 = \frac{1}{3}$ . The Green function associated with L is therefore given by

$$g(t) = \frac{1}{3} e^{-2t} \sin(3t).$$

**Laplace.** The characteristic polynomial of L is  $p(z) = z^2 + 4z + 13 = (z+2)^2 + 3^2$ . The Green function associated with L is then given by

$$g(t) = \mathcal{L}^{-1}\left(\frac{1}{p(s)}\right) = \mathcal{L}^{-1}\left(\frac{1}{(s+2)^2 + 3^2}\right).$$

Referring to the table on the last page, item 3 with  $a = -2$  and  $b = 3$  gives

$$g(t) = \frac{1}{3} \mathcal{L}^{-1}\left(\frac{3}{(s+2)^2 + 3^2}\right) = \frac{1}{3} e^{-2t} \sin(3t).$$

- (b) Use the Green function to express a particular solution  $Y_P(t)$  in terms of two definite integrals. DO NOT evaluate the definite integrals.

**Solution.** For any initial time  $t_I$  a particular solution  $Y_P(t)$  is given by

$$Y_P(t) = \int_{t_I}^t g(t-s) \sin(s^2) ds = \frac{1}{4} \int_{t_I}^t e^{-2(t-s)} \sin(3(t-s)) \sin(s^2) ds.$$

Because  $\sin(3(t-s)) = \sin(3t) \cos(3s) - \cos(3t) \sin(3s)$ , this particular solution is given in terms of definite integrals as

$$\begin{aligned} Y_P(t) &= \frac{1}{3} e^{-2t} \sin(3t) \int_{t_I}^t e^{2s} \cos(3s) \sin(s^2) ds \\ &\quad - \frac{1}{3} e^{-2t} \cos(3t) \int_{t_I}^t e^{2s} \sin(3s) \sin(s^2) ds. \end{aligned}$$

**Remark:** The above definite integrals cannot be evaluated analytically.

- (4) [12] The functions  $e^{\frac{1}{3}t^3}$  and  $te^{\frac{1}{3}t^3}$  are solutions of the homogeneous equation

$$\frac{d^2 y}{dt^2} - 2t^2 \frac{dy}{dt} + (t^4 - 2t)y = 0.$$

(You do not have to check that this is true!)

- (a) Compute their Wronskian.

**Solution.** The Wronskian is

$$\begin{aligned} W[e^{\frac{1}{3}t^3}, te^{\frac{1}{3}t^3}](t) &= \det \begin{pmatrix} e^{\frac{1}{3}t^3} & te^{\frac{1}{3}t^3} \\ t^2 e^{\frac{1}{3}t^3} & (t^3 + 1)e^{\frac{1}{3}t^3} \end{pmatrix} \\ &= e^{\frac{1}{3}t^3} \cdot (t^3 + 1)e^{\frac{1}{3}t^3} - t^2 e^{\frac{1}{3}t^3} \cdot te^{\frac{1}{3}t^3} = e^{\frac{2}{3}t^3}. \end{aligned}$$

(b) Solve the initial-value problem

$$\frac{d^2y}{dt^2} - 2t^2 \frac{dy}{dt} + (t^4 - 2t)y = t^2 e^{\frac{1}{3}t^3}, \quad y(0) = y'(0) = 0.$$

Try to evaluate all integrals explicitly.

**Solution.** This nonhomogeneous equation has variable coefficients, so you must use either the general Green function method or the variation of parameters method to solve it. It is already in normal form. Because by part (a)

$$W[e^{\frac{1}{3}t^3}, te^{\frac{1}{3}t^3}](t) = e^{\frac{2}{3}t^3} \neq 0,$$

you know that  $e^{\frac{1}{3}t^3}$  and  $te^{\frac{1}{3}t^3}$  constitute a fundamental set of solutions to the associated homogeneous equation.

**General Green Function.** The Green function  $G(t, s)$  is given by

$$\begin{aligned} G(t, s) &= \frac{\det \begin{pmatrix} e^{\frac{1}{3}s^3} & se^{\frac{1}{3}s^3} \\ e^{\frac{1}{3}t^3} & te^{\frac{1}{3}t^3} \end{pmatrix}}{W[e^{\frac{1}{3}s^3}, se^{\frac{1}{3}s^3}](s)} = \frac{e^{\frac{1}{3}s^3} te^{\frac{1}{3}t^3} - e^{\frac{1}{3}t^3} se^{\frac{1}{3}s^3}}{e^{\frac{2}{3}t^3}} \\ &= (t - s) \frac{e^{\frac{1}{3}t^3}}{e^{\frac{1}{3}s^3}} = (t - s) e^{\frac{1}{3}t^3} e^{-\frac{1}{3}t^3}. \end{aligned}$$

The Green function formula then yields the solution

$$\begin{aligned} y(t) &= \int_0^t G(t, s) s^2 e^{\frac{1}{3}s^3} ds = e^{\frac{1}{3}t^3} \int_0^t (t - s) s^2 ds \\ &= te^{\frac{1}{3}t^3} \int_0^t s^2 ds - e^{\frac{1}{3}t^3} \int_0^t s^3 ds \\ &= te^{\frac{1}{3}t^3} \frac{1}{3}t^3 - e^{\frac{1}{3}t^3} \frac{1}{4}t^4 = \frac{1}{12}t^4 e^{\frac{1}{3}t^3}. \end{aligned}$$

**Variation of Parameters.** A general solution of the associated homogeneous equation is

$$y_H(t) = c_1 e^{\frac{1}{3}t^3} + c_2 t e^{\frac{1}{3}t^3}.$$

Seek a solution in the form

$$y = u_1(t) e^{\frac{1}{3}t^3} + u_2(t) t e^{\frac{1}{3}t^3},$$

where  $u_1'(t)$  and  $u_2'(t)$  satisfy the linear algebraic system

$$\begin{aligned} u_1'(t) e^{\frac{1}{3}t^3} + u_2'(t) t e^{\frac{1}{3}t^3} &= 0, \\ u_1'(t) t^2 e^{\frac{1}{3}t^3} + u_2'(t) (t^3 + 1) e^{\frac{1}{3}t^3} &= t^2 e^{\frac{1}{3}t^3}. \end{aligned}$$

Solve this system to obtain the explicit first-order equations

$$u_1'(t) = -t^3, \quad u_2'(t) = t^2.$$

Integrate these equations to find

$$u_1(t) = c_1 - \frac{1}{4}t^4, \quad u_2(t) = c_2 + \frac{1}{3}t^3.$$

A general solution is therefore

$$\begin{aligned} y(t) &= c_1 e^{\frac{1}{3}t^3} + c_2 t e^{\frac{1}{3}t^3} - \frac{1}{4}t^4 e^{\frac{1}{3}t^3} + \frac{1}{3}t^3 t e^{\frac{1}{3}t^3} \\ &= c_1 e^{\frac{1}{3}t^3} + c_2 t e^{\frac{1}{3}t^3} + \frac{1}{12}t^4 e^{\frac{1}{3}t^3}. \end{aligned}$$

Because

$$y'(t) = c_1 t^2 e^{\frac{1}{3}t^3} + c_2 (t^3 + 1) e^{\frac{1}{3}t^3} + \frac{1}{12}(t^6 + 4t^3) e^{\frac{1}{3}t^3},$$

when the initial conditions are imposed you find that

$$y(0) = c_1 \cdot 1 + c_2 \cdot 0 + 0 = 0, \quad y'(0) = c_1 \cdot 0 + c_2 \cdot 1 + 0 = 0.$$

These show that  $c_1 = c_2 = 0$ . The solution of the initial-value problem is therefore

$$y(t) = \frac{1}{12}t^4 e^{\frac{1}{3}t^3}.$$

- (5) [4] What answer will be produced by the following MATLAB commands?

```
>> ode1 = 'D2y - 2*Dy + 5*y = 0';
>> dsolve(ode1, 't')
ans =
```

**Solution.** The commands ask MATLAB to give a general solution of the equation

$$D^2y - 2Dy + 5y = 0, \quad \text{where } D = \frac{d}{dt}.$$

MATLAB will produce the answer

$$C1*\exp(t)*\sin(2*t) + C2*\exp(t)*\cos(2*t)$$

This can be seen as follows. This is a constant coefficient, homogeneous equation. The characteristic polynomial is

$$p(z) = z^2 - 2z + 5 = (z - 1)^2 + 4 = (z - 1)^2 + 2^2.$$

Its roots are the conjugate pair  $-1 \pm i2$ . A general solution is therefore

$$y = c_1 e^t \cos(2t) + c_2 e^t \sin(2t).$$

Up to notational differences, this is the answer that MATLAB produces.

- (6) [8] Solve the initial-value problem

$$y'' + 4y = 9 \sin(t), \quad y(0) = 0, \quad y'(0) = 5.$$

**Solution.** This is a constant coefficient, nonhomogeneous equation. Its characteristic polynomial is

$$p(z) = z^2 + 4 = z^2 + 2^2.$$

This has the conjugate pair of roots  $\pm i2$ . A general solution of the associated homogeneous equation is

$$y_H(t) = c_1 \cos(2t) + c_2 \sin(2t).$$

The forcing  $9 \sin(t)$  has degree  $d = 0$  and characteristic  $r + is = i$ , which is a root of  $p(z)$  of multiplicity  $m = 0$ . A particular solution  $y_P(t)$  can be found by the method of

undetermined coefficients using either KEY identity evaluation or direct substitution. You could also solve the initial-value problem using the Laplace transform.

**KEY Identity Evaluation.** Because  $m + d = 0$ , you only need to evaluate the KEY identity at  $z = i$ , to find

$$L(e^t) = p(i)e^{it} = (i^2 + 4)e^{it} = (-1 + 4)e^{it} = 3e^{it}.$$

Multiplying this equation by 3 yields  $L(3e^{it}) = 9e^{it}$ . Taking the imaginary part of both sides then gives  $L(3\sin(t)) = 9\sin(t)$ . Hence,  $y_P(t) = 3\sin(t)$ .

**Direct Substitution.** Because  $m = d = 0$ , you seek a particular solution of the form

$$y_P(t) = A\cos(t) + B\sin(t).$$

Because

$$y'_P(t) = -A\sin(t) + B\cos(t), \quad y''_P(t) = -A\cos(t) - B\sin(t),$$

one sees that

$$\begin{aligned} Ly_P(t) = y''_P(t) + 4y_P(t) &= -A\cos(t) - B\sin(t) + 4A\cos(t) + 4B\sin(t) \\ &= 3A\cos(t) + 3B\sin(t). \end{aligned}$$

Setting  $Ly_P(t) = 3A\cos(t) + 3B\sin(t) = 9\sin(t)$ , we see that  $A = 0$  and  $B = 3$ . Hence,  $y_P(t) = 3\sin(t)$ .

By either approach you find that  $y_P(t) = 3\sin(t)$ , which yields the general solution

$$y(t) = c_1\cos(2t) + c_2\sin(2t) + 3\sin(t).$$

Because

$$y'(t) = -2c_1\sin(2t) + 2c_2\cos(2t) + 3\cos(t).$$

when the initial conditions are imposed you obtain

$$y(0) = c_1 \cdot 1 + c_2 \cdot 0 + 0 = 0, \quad y'(0) = -2c_1 \cdot 0 + 2c_2 \cdot 1 + 3 = 5.$$

These are solved to find that  $c_1 = 0$  and  $c_2 = 1$ . The solution of the initial-value problem is therefore

$$y(t) = \sin(2t) + 3\sin(t).$$

**Laplace.** The Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) + 4\mathcal{L}[y](s) = 9\mathcal{L}[\sin(t)](s).$$

If we set  $\mathcal{L}[y](s) = Y(s)$  then

$$\begin{aligned} \mathcal{L}[y'](s) &= sY(s) - y(0) = sY(s), \\ \mathcal{L}[y''](s) &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 5. \end{aligned}$$

Referring to the table on the last page, item 3 with  $a = 0$  and  $b = 1$  yields

$$\mathcal{L}[\sin(t)](s) = \frac{1}{s^2 + 1}.$$

The Laplace transform of the initial-value problem thereby becomes

$$(s^2 + 4)Y(s) - 5 = \frac{9}{s^2 + 1}.$$

This can be solved for  $Y(s)$  to obtain

$$Y(s) = \frac{5}{s^2 + 4} + \frac{9}{(s^2 + 1)(s^2 + 4)}.$$

The partial fraction identity

$$\frac{9}{(z + 1)(z + 4)} = \frac{3}{z + 1} + \frac{-3}{z + 4} \quad \text{evaluated at } z = s^2$$

yields the partial fraction identity

$$\frac{9}{(s^2 + 1)(s^2 + 4)} = \frac{3}{s^2 + 1} - \frac{3}{s^2 + 4}.$$

It follows that

$$Y(s) = \frac{5}{s^2 + 4} + \frac{3}{s^2 + 1} - \frac{3}{s^2 + 4} = \frac{2}{s^2 + 4} + \frac{3}{s^2 + 1}.$$

Referring to the table on the last page, item 3 with  $a = 0$  and  $b = 2$ , and with  $a = 0$  and  $b = 1$  shows that the solution to the initial-value problem is

$$y(t) = \mathcal{L}^{-1}[Y(s)](t) = \mathcal{L}^{-1}\left[\frac{2}{s^2 + 2^2}\right](t) + 3\mathcal{L}^{-1}\left[\frac{1}{s^2 + 1^2}\right](t) = \sin(2t) + 3\sin(t).$$

(7) [8] Give a general real solution of the equation

$$y'' - y = e^t.$$

**Solution.** This is a constant coefficient, nonhomogeneous equation. Its characteristic polynomial is

$$p(z) = z^2 - 1 = (z + 1)(z - 1).$$

This has the real roots  $-1$  and  $1$ . A general solution of the associated homogeneous equation is

$$y_H(t) = c_1 e^{-t} + c_2 e^t.$$

The forcing  $e^t$  has degree  $d = 0$  and characteristic  $r + is = 1$ , which is a root of  $p(z)$  of multiplicity  $m = 1$ . A particular solution  $y_P(t)$  can be found by the method of undetermined coefficients using either KEY identity evaluation or direct substitution.

**KEY Identity Evaluation.** Because  $m + d = 1$ , you need the KEY identity and its first derivative

$$\begin{aligned} \mathcal{L}(e^{zt}) &= (z^2 - 1)e^{zt}, \\ \mathcal{L}(t e^{zt}) &= (z^2 - 1)t e^{zt} + 2z e^{zt}. \end{aligned}$$

Evaluate these at  $z = 1$  to find

$$\mathcal{L}(e^t) = 0, \quad \mathcal{L}(t e^t) = 2e^t.$$

Dividing the second equation by 2 yields  $\mathcal{L}(\frac{1}{2}t e^t) = e^t$ . Hence,  $y_P(t) = \frac{1}{2}t e^t$ . A general solution is therefore

$$y = c_1 e^{-t} + c_2 e^t + \frac{1}{2}t e^t.$$

**Direct Substitution.** Because  $m = 1$  and  $d = 0$ , you seek a particular solution of the form

$$y_P(t) = At e^t,$$

Because

$$\begin{aligned} y'_P(t) &= At e^t + A e^t = A(t+1) e^t, \\ y''_P(t) &= A(t+1) e^t + A e^t = A(t+2) e^t, \end{aligned}$$

one sees that

$$Ly_P(t) = y''_P(t) - y_P(t) = A(t+2) e^t - At e^t = 2A e^t.$$

Setting  $Ly_P(t) = 2A e^t = e^t$ , we see that  $2A = 1$ . It follows that  $A = \frac{1}{2}$ , whereby  $y_P(t) = \frac{1}{4}t e^t$ . A general solution is therefore

$$y = c_1 e^{-t} + c_2 e^t + \frac{1}{2}t e^t.$$

(8) [8] The vertical displacement of a mass on a spring is given by

$$h(t) = 3e^{-t} \cos(\pi t) - 4e^{-t} \sin(\pi t).$$

Express this in the form  $h(t) = Ae^{-t} \cos(\pi t - \delta)$  with  $A > 0$  and  $0 \leq \delta < 2\pi$ , identifying the period and phase of the oscillation. (The phase may be expressed in terms of an inverse trig function.)

**Solution.** By comparing

$$Ae^{-t} \cos(\pi t - \delta) = Ae^{-t} \cos(\delta) \cos(\pi t) + Ae^{-t} \sin(\delta) \sin(\pi t),$$

with  $h(t) = 3e^{-t} \cos(\pi t) - 4e^{-t} \sin(\pi t)$ , we see that

$$A \cos(\delta) = 3, \quad A \sin(\delta) = -4.$$

This shows that  $(A, \delta)$  are the polar coordinates of the point in the plane whose Cartesian coordinates are  $(3, -4)$ . Clearly  $A$  is given by

$$A = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5.$$

Because  $(3, -4)$  lies in the fourth quadrant, the phase  $\delta$  satisfies  $\frac{3\pi}{2} < \delta < 2\pi$ . Because

$$\cos(\delta) = \frac{3}{5}, \quad \sin(\delta) = -\frac{4}{5}, \quad \tan(\delta) = -\frac{4}{3},$$

you can express the phase by any one of the formulas

$$\delta = 2\pi - \cos^{-1}\left(\frac{3}{5}\right), \quad \delta = 2\pi - \sin^{-1}\left(\frac{4}{5}\right), \quad \delta = 2\pi - \tan^{-1}\left(\frac{4}{3}\right).$$

Because the quasi frequency  $\nu$  is given by  $\nu = \pi$ , the quasi period  $T$  is given by

$$T = \frac{2\pi}{\nu} = \frac{2\pi}{\pi} = 2.$$

(9) [8] When a mass of 2 kilograms is hung vertically from a spring, at rest it stretches the spring .2 m. (Gravitational acceleration is  $g = 9.8 \text{ m/sec}^2$ .) At  $t = 0$  the mass is displaced .1 m above its rest position and is released with a downward initial velocity of .3 m/sec. Assume that the spring force is proportional to displacement, that there is no drag force, and that the mass is driven by an external force of  $F_{ext}(t) = 10 \cos(\omega t)$  Newtons (1 Newton = 1 kg m/sec<sup>2</sup>), where up is taken to be positive.

- (a) Formulate an initial-value problem that governs the motion of the mass for  $t > 0$ . DO NOT solve this initial-value problem, just write it down!

**Solution.** Let  $h(t)$  be the displacement (in meters) of the mass from its equilibrium (rest) position at time  $t$  (in seconds), with upward displacements being positive. The governing initial-value problem then has the form

$$m \frac{d^2 h}{dt^2} + kh = F_{ext}(t), \quad h(0) = .1, \quad h'(0) = -.3,$$

where  $m$  is the mass and  $k$  is the spring constant. The problem says that  $m = 2$  kilograms. The spring constant is obtained by balancing the weight of the mass ( $mg = 2 \cdot 9.8$  Newtons) with the force applied by the spring when it is stretched .2 m. This gives  $k \cdot .2 = 2 \cdot 9.8$ , or

$$k = \frac{2 \cdot 9.8}{.2} = \frac{2 \cdot 98}{2} = 98 \text{ Newtons/m.}$$

Because  $F_{ext}(t) = 10 \cos(\omega t)$ , the governing initial-value problem is therefore

$$2 \frac{d^2 h}{dt^2} + 98 h = 10 \cos(\omega t), \quad h(0) = -.1, \quad h'(0) = -.3.$$

- (b) What is the natural frequency of this spring?

**Solution.** The natural frequency  $\omega_o$  of the spring is given by

$$\omega_o = \sqrt{\frac{k}{m}} = \sqrt{\frac{98}{2}} = \sqrt{49} = 7 \text{ 1/sec (or rad/sec).}$$

- (c) At what value of the driving frequency  $\omega$  does resonance occur?

**Solution.** Resonance occurs when the driving frequency  $\omega$  equals the natural frequency  $\omega_o$ . Hence, by the answer to part (b), resonance occurs when

$$\omega = \omega_o = 7 \text{ 1/sec (or rad/sec).}$$

- (10) [8] Compute the Laplace transform of  $f(t) = u(t-3)e^{-2t}$  from its definition. (Here  $u$  is the unit step function.)

**Solution.** The definition of Laplace transform gives

$$\mathcal{L}[f](s) = \lim_{T \rightarrow \infty} \int_0^T e^{-st} u(t-3) e^{-2t} dt = \lim_{T \rightarrow \infty} \int_3^T e^{-(s+2)t} dt.$$

This limit diverges to  $+\infty$  for  $s \leq -2$  because in that case

$$\int_3^T e^{-(s+2)t} dt \geq \int_3^T 1 dt = T - 3,$$

which clearly diverges to  $+\infty$  as  $T \rightarrow \infty$ .

For  $s > -2$  you obtain

$$\int_3^T e^{-(s+2)t} dt = -\frac{e^{-(s+2)t}}{s+2} \Big|_3^T = -\frac{e^{-(s+2)T}}{s+2} + \frac{e^{-(s+2)3}}{s+2}.$$



Hence, for  $s > -2$  one has that

$$\mathcal{L}[f](s) = \lim_{T \rightarrow \infty} \int_3^T e^{-(s+2)t} dt = \lim_{T \rightarrow \infty} \left( \frac{e^{-(s+2)3}}{s+2} - \frac{e^{-(s+2)T}}{s+2} \right) = \frac{e^{-(s+2)3}}{s+2}.$$

(11) [8] Find the Laplace transform  $Y(s)$  of the solution  $y(t)$  of the initial-value problem

$$\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 20y = f(t), \quad y(0) = 5, \quad y'(0) = -3,$$

where

$$f(t) = \begin{cases} t & \text{for } 0 \leq t < 3, \\ 3e^{-(t-3)} & \text{for } 3 \leq t. \end{cases}$$

You may refer to the table on the last page. DO NOT take the inverse Laplace transform to find  $y(t)$ , just solve for  $Y(s)$ !

**Solution.** The Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) + 8\mathcal{L}[y'](s) + 20\mathcal{L}[y](s) = \mathcal{L}[f](s).$$

If we set  $\mathcal{L}[y](s) = Y(s)$  then

$$\mathcal{L}[y'](s) = sY(s) - y(0) = sY(s) - 5,$$

$$\mathcal{L}[y''](s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 5s + 3.$$

To compute  $\mathcal{L}[f](s)$  you first express  $f(t)$  in terms of unit step functions as

$$\begin{aligned} f(t) &= t(u(t) - u(t-3)) + 3e^{-(t-3)}u(t-3) \\ &= u(t)t + u(t-3)(3e^{-(t-3)} - t) \\ &= t + u(t-3)(3e^{-(t-3)} - (t-3) - 3). \end{aligned}$$

Referring to the table on the last page, item 1 with  $a = 0$  and  $n = 1$ , item 6 with  $c = 3$  and  $f(t) = 3e^{-t} - t - 3$ , and item 1 with  $a = -1$  and  $n = 0$ , and with  $a = 0$  and  $n = 0$  yields

$$\begin{aligned} \mathcal{L}[f](s) &= \mathcal{L}[t](s) + \mathcal{L}[u(t-3)(3e^{-(t-3)} - (t-3) - 3)](s) \\ &= \frac{1}{s^2} + e^{-3s}\mathcal{L}[3e^{-t} - t - 3](s) \\ &= \frac{1}{s^2} + e^{-3s}\left(\frac{3}{s+1} - \frac{1}{s^2} - \frac{3}{s}\right). \end{aligned}$$

The Laplace transform of the initial-value problem then becomes

$$(s^2Y(s) - 5s + 3) + 8(sY(s) - 5) + 20Y(s) = \frac{1}{s^2} + e^{-3s}\left(\frac{3}{s+1} - \frac{1}{s^2} - \frac{3}{s}\right),$$

which becomes

$$(s^2 + 8s + 20)Y(s) - 5s + 3 - 40 = \frac{1}{s^2} + e^{-3s}\left(\frac{3}{s+1} - \frac{1}{s^2} - \frac{3}{s}\right).$$

Hence,  $Y(s)$  is given by

$$Y(s) = \frac{1}{s^2 + 8s + 20} \left( 5s + 37 + \frac{1}{s^2} + e^{-3s} \left( \frac{3}{s+1} - \frac{1}{s^2} - \frac{3}{s} \right) \right).$$

- (12) [16] Find the inverse Laplace transforms of the following functions. You may refer to the table on the last page.

(a)  $F(s) = \frac{8}{s^2 - 2s - 3},$

**Solution.** The denominator factors as  $(s - 3)(s + 1)$ , so the partial fraction decomposition is

$$\frac{8}{s^2 - 2s - 3} = \frac{8}{(s - 3)(s + 1)} = \frac{2}{s - 3} + \frac{-2}{s + 1}.$$

Referring to the table on the last page, item 1 with  $a = 3$  and  $n = 0$  and with  $a = -1$  and  $n = 0$  gives

$$\begin{aligned} \mathcal{L}[e^{3t}](s) &= \frac{1}{s - 3}, & \implies & \mathcal{L}^{-1}\left[\frac{1}{s - 3}\right](t) = e^{3t}, \\ \mathcal{L}[e^{-t}](s) &= \frac{1}{s + 1}, & \implies & \mathcal{L}^{-1}\left[\frac{1}{s + 1}\right](t) = e^{-t}. \end{aligned}$$

By the linearity of  $\mathcal{L}^{-1}$  you therefore conclude that

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{8}{s^2 - 2s - 3}\right](t) &= \mathcal{L}^{-1}\left[\frac{2}{s - 3} - \frac{2}{s + 1}\right](t) \\ &= 2\mathcal{L}^{-1}\left[\frac{1}{s - 3}\right](t) - 2\mathcal{L}^{-1}\left[\frac{1}{s + 1}\right](t) \\ &= 2e^{3t} - 2e^{-t}. \end{aligned}$$

(b)  $F(s) = \frac{8e^{-\pi s}}{s^2 - 6s + 13}.$

**Solution.** Complete the square in the denominator to get  $(s - 3)^2 + 4$ . Referring to the table on the last page, item 3 with  $a = 3$  and  $b = 2$  gives

$$\mathcal{L}[e^{3t} \sin(2t)](s) = \frac{2}{(s - 3)^2 + 2^2}, \implies \mathcal{L}^{-1}\left[\frac{2}{s^2 - 6s + 13}\right](t) = e^{3t} \sin(2t).$$

By item 6 with  $c = \pi$  and  $f(t) = 4e^{3t} \sin(2t)$  you therefore conclude that

$$\begin{aligned} \mathcal{L}^{-1}\left[e^{-\pi s} \frac{8}{s^2 - 6s + 13}\right](t) &= u(t - \pi) \mathcal{L}^{-1}\left[\frac{8}{s^2 - 6s + 13}\right](t - \pi) \\ &= u(t - \pi) 4e^{3(t - \pi)} \sin(2(t - \pi)) \\ &= u(t - \pi) 4e^{3(t - \pi)} \sin(2t). \end{aligned}$$

### A Short Table of Laplace Transforms

$$\mathcal{L}[e^{at}t^n](s) = \frac{n!}{(s-a)^{n+1}} \quad \text{for } s > a,$$

$$\mathcal{L}[e^{at}\cos(bt)](s) = \frac{s-a}{(s-a)^2+b^2} \quad \text{for } s > a,$$

$$\mathcal{L}[e^{at}\sin(bt)](s) = \frac{b}{(s-a)^2+b^2} \quad \text{for } s > a,$$

$$\mathcal{L}[e^{at}f(t)](s) = F(s-a) \quad \text{where } F(s) = \mathcal{L}[f(t)](s),$$

$$\mathcal{L}[t^n f(t)](s) = (-1)^n F^{(n)}(s) \quad \text{where } F(s) = \mathcal{L}[f(t)](s),$$

$$\mathcal{L}[u(t-c)f(t-c)](s) = e^{-cs}F(s) \quad \text{where } F(s) = \mathcal{L}[f(t)](s) \\ \text{and } u \text{ is the step function.}$$