Third In-Class Exam Solutions
Math 246, Spring 2009, Professor David Levermore
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(1) [8] Consider the matrices

$$\mathbf{A} = \begin{pmatrix} 2 & -4 \\ 1 & 3 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}.$$

Compute the matrices

(a) $\mathbf{AB}$ **Solution.** $\mathbf{AB} = \begin{pmatrix} 2 & -4 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} -2 & -6 \\ 9 & 17 \end{pmatrix}$

(b) $\mathbf{B}^{-1}$ **Solution.** Because $\det(\mathbf{B}) = 3 \cdot 4 - 2 \cdot 5 = 12 - 10 = 2$,

$$\mathbf{B}^{-1} = \frac{1}{\det(\mathbf{B})} \begin{pmatrix} 4 & -5 \\ -2 & 3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 4 & -5 \\ -2 & 3 \end{pmatrix}.$$

(2) [15] Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 5 & 2 \\ 8 & -1 \end{pmatrix}.$$

(a) Find all the eigenvalues of $\mathbf{A}$.

Solution. The characteristic polynomial of $\mathbf{A}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 4z - 21 = (z+3)(z-7).$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are -3 and 7.

(b) For each eigenvalue of $\mathbf{A}$ find an associated eigenvector.

Solution (using the Cayley-Hamilton method from notes). One has

$$\mathbf{A} + 3\mathbf{I} = \begin{pmatrix} 8 & 2 \\ 8 & 2 \end{pmatrix}, \qquad \mathbf{A} - 7\mathbf{I} = \begin{pmatrix} -2 & 2 \\ 8 & -8 \end{pmatrix}.$$

Every nonzero column of $\mathbf{A} - 7\mathbf{I}$ has the form

$$\alpha_1 \begin{pmatrix} -1 \\ 4 \end{pmatrix} \quad \text{for some } \alpha_1 \neq 0,$$

any of which is an eigenvector associated with -3. Similarly, every nonzero column of $\mathbf{A} + 3\mathbf{I}$ has the form

$$\alpha_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{for some } \alpha_2 \neq 0,$$

any of which is an eigenvector associated with 7.

(c) Diagonalize $\mathbf{A}$.

Solution. Because $\mathbf{A}$ has the eigenpairs

$$\left(7, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right), \qquad \left(-3, \begin{pmatrix} -1 \\ 4 \end{pmatrix}\right),$$

set

$$\mathbf{V} = \begin{pmatrix} 1 & -1 \\ 1 & 4 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 7 & 0 \\ 0 & -3 \end{pmatrix}.$$

Because $\det(\mathbf{V}) = 4 - (-1) = 5$,

$$\mathbf{V}^{-1} = \frac{1}{\det(\mathbf{V})} \begin{pmatrix} 4 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 4 & 1 \\ -1 & 1 \end{pmatrix}.$$

Then $\mathbf{A}$ has the diagonalization

$$\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 7 & 0 \\ 0 & -3 \end{pmatrix} \frac{1}{5} \begin{pmatrix} 4 & 1 \\ -1 & 1 \end{pmatrix}.$$

(3) [10] Suppose you know that $e^{t\mathbf{A}} = \begin{pmatrix} \cos(2t) + \sin(2t) & -\sin(2t) \\ 2\sin(2t) & \cos(2t) - \sin(2t) \end{pmatrix}$.

(a) Solve the initial-value problem

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Solution. The solution is given by

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{t\mathbf{A}} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} &= \begin{pmatrix} \cos(2t) + \sin(2t) & -\sin(2t) \\ 2\sin(2t) & \cos(2t) - \sin(2t) \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} \cos(2t) - \sin(2t) \\ 2\cos(2t) \end{pmatrix}. \end{aligned}$$

(b) Determine $\mathbf{A}$.

Solution. The simplest way to do this is

$$\mathbf{A} = \left.\frac{\mathrm{d}e^{t\mathbf{A}}}{\mathrm{d}t}\right|_{t=0} = \left.\begin{pmatrix} -2\sin(2t) + 2\cos(2t) & -2\cos(2t) \\ 4\cos(2t) & -2\sin(2t) - 2\cos(2t) \end{pmatrix}\right|_{t=0} = \begin{pmatrix} 2 & -2 \\ 4 & -2 \end{pmatrix}.$$

Alternative Solution. Because $\dfrac{\mathrm{d}e^{t\mathbf{A}}}{\mathrm{d}t} = \mathbf{A}e^{t\mathbf{A}}$, and because $\left(e^{t\mathbf{A}}\right)^{-1} = e^{-t\mathbf{A}}$, you see that

$$\mathbf{A} = \frac{\mathrm{d}e^{t\mathbf{A}}}{\mathrm{d}t} \left(e^{t\mathbf{A}}\right)^{-1} = \frac{\mathrm{d}e^{t\mathbf{A}}}{\mathrm{d}t} e^{-t\mathbf{A}}.$$

Because $\mathbf{A}$ is independent of t you may evaluate the right-hand side at any t. It is best to set $t = 0$ on the right-hand side because $e^{0\mathbf{A}} = \mathbf{I}$. The right-hand side is then evaluated as in the previous solution.

(4) [10] Consider two interconnected tanks filled with brine (salt water). The first tank contains 70 liters and the second contains 40 liters. Brine flows with a concentration of 3 grams of salt per liter flows into the first tank at a rate of 5 liters per hour. Well stirred brine flows from the first tank to the second at a rate of 7 liters per hour, from the second to the first at a rate of 2 liters per hour, and from the second into a drain at a rate of 5 liters per hour. At $t = 0$ there are 35 grams of salt in the first tank and 25 grams in the second. Give an initial-value problem that governs the amount of salt in each tank as a function of time.

Solution. The rates work out so there will always be 70 liters of brine in the first tank and 40 liters in the second. Let $S_1(t)$ and $S_2(t)$ be the grams of salt in the first and second tanks respectively. These are governed by the initial-value problem

$$\begin{aligned} \frac{\mathrm{d}S_1}{\mathrm{d}t} &= 3\cdot 5 + \frac{S_2}{40}\,2 - \frac{S_1}{70}\,7\,, \qquad & S_1(0) = 35\,, \\ \frac{\mathrm{d}S_2}{\mathrm{d}t} &= \frac{S_1}{70}\,7 - \frac{S_2}{40}\,2 - \frac{S_2}{40}\,5\,, \qquad & S_2(0) = 25\,. \end{aligned}$$

(5) [8] Transform the equation $\dfrac{\mathrm{d}^4y}{\mathrm{d}t^4} + e^t\dfrac{\mathrm{d}^3y}{\mathrm{d}t^3} - \dfrac{\mathrm{d}y}{\mathrm{d}t} + 5y = t^2$ into a first-order system of ordinary differential equations.

Solution: Because the equation is fourth order, the first order system must have dimension four. The simplest such first order system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ t^2 - 5x_1 + x_2 - e^t x_4 \end{pmatrix}, \qquad \text{where} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} y \\ y' \\ y'' \\ y''' \end{pmatrix}.$$

(6) [15] Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} 1+t^5 \\ 2t^2 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t^3 \\ 2 \end{pmatrix}$.

(a) Compute the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2](t)$.

Solution.

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \det\begin{pmatrix} 1+t^5 & t^3 \\ 2t^2 & 2 \end{pmatrix} = (1+t^5)2 - 2t^5 = 2\,.$$

(b) Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to the system $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x}$ wherever $W[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$.

Solution. Let $\mathbf{\Psi}(t) = \begin{pmatrix} 1+t^5 & t^3 \\ 2t^2 & 2 \end{pmatrix}$. Because $\dfrac{\mathrm{d}\mathbf{\Psi}(t)}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{\Psi}(t)$, one has

$$\begin{aligned} \mathbf{A}(t) &= \frac{\mathbf{\Psi}(t)}{\mathrm{d}t}\,\mathbf{\Psi}(t)^{-1} = \begin{pmatrix} 5t^4 & 3t^2 \\ 4t & 0 \end{pmatrix}\begin{pmatrix} 1+t^5 & t^3 \\ 2t^2 & 2 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 5t^4 & 3t^2 \\ 4t & 0 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 2 & -t^3 \\ -2t^2 & 1+t^5 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4t^4 & 3t^3 - 2t^7 \\ 8t & -4t^4 \end{pmatrix} = \begin{pmatrix} 2t^4 & \frac{3}{2}t^2 - t^7 \\ 4t & -2t^4 \end{pmatrix}. \end{aligned}$$

(c) Give a general solution to the system you found in part (b).

Solution. Because $W[\mathbf{x}_1, \mathbf{x}_2](t) = 2 \neq 0$, a general solution is

$$\mathbf{x} = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1 \begin{pmatrix} 1+t^5 \\ 2t^2 \end{pmatrix} + c_2 \begin{pmatrix} t^3 \\ 2 \end{pmatrix}.$$

(7) [16] Find a general solution for each of the following systems.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 & 4 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Solution. Let $\mathbf{A} = \begin{pmatrix} -3 & 4 \\ -1 & 1 \end{pmatrix}$. The characteristic polynomial of $\mathbf{A}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 2z + 1 = (z+1)^2,$$

which has the double root -1. Then, because $\mu = -1$ and $\nu = 0$,

$$\begin{aligned} e^{t\mathbf{A}} = e^{-t}\big[\mathbf{I} + (\mathbf{A}+\mathbf{I})\,t\big] &= e^{-t}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix} t\right] \\ &= e^{-t}\begin{pmatrix} 1-2t & 4t \\ -t & 1+2t \end{pmatrix}. \end{aligned}$$

A general solution is therefore

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{t\mathbf{A}}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1 e^{-t}\begin{pmatrix} 1-2t \\ -t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 4t \\ 1+2t \end{pmatrix}.$$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Solution. Let $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 5 & -1 \end{pmatrix}$. The characteristic polynomial of $\mathbf{A}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 9 = z^2 + 3^2,$$

which has the conjugate pair of roots $\pm i3$. Then, because $\mu = 0$ and $\nu = 4$,

$$\begin{aligned} e^{t\mathbf{A}} = \mathbf{I}\cos(3t) + \mathbf{A}\frac{\sin(3t)}{3} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(3t) + \begin{pmatrix} 1 & -2 \\ 5 & -1 \end{pmatrix}\frac{\sin(3t)}{3} \\ &= \begin{pmatrix} \cos(3t) + \frac{1}{3}\sin(3t) & -\frac{2}{3}\sin(4t) \\ \frac{5}{3}\sin(3t) & \cos(3t) - \frac{1}{3}\sin(3t) \end{pmatrix}. \end{aligned}$$

A general solution is therefore

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{t\mathbf{A}}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1\begin{pmatrix} \cos(3t) + \frac{1}{3}\sin(3t) \\ \frac{5}{3}\sin(3t) \end{pmatrix} + c_2\begin{pmatrix} -\frac{2}{3}\sin(3t) \\ \cos(3t) - \frac{1}{3}\sin(3t) \end{pmatrix}.$$

(8) [10] Sketch the phase-plane portrait for each of the two systems in the previous problem. For each portrait identify its type and give a reason why the origin is either attracting, stable, unstable, or repelling.

Solution (a). The coefficient matrix $\mathbf{A}$ has the eigenvalue -1. Because

$$\mathbf{A} + \mathbf{I} = \begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix},$$

it has the eigenpair

$$\left(-1 \,, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right).$$

Because $\mathbf{A} \neq -\mathbf{I}$, the portrait is a *twist sink* (*improper nodal sink*) and is thereby *attracting* (*asymptotically stable*). Because $a_{21} = -1 < 0$, the phase portrait is a *clockwise twist sink*. There is one trajectory that approaches the origin along each half of the line $y = \frac{1}{2}x$. Trajectories above the line $y = \frac{1}{2}x$ will approach the origin tangent to the line $y = \frac{1}{2}x$ from the right. Trajectories below the line $y = \frac{1}{2}x$ will approach the origin tangent to the line $y = \frac{1}{2}x$ from the left.

Solution (b). The coefficient matrix $\mathbf{A}$ has the eigenvalues $\pm i3$. The portrait is therefore a *center* and the origin is thereby *stable*. Because $a_{21} = 5 > 0$, the phase portrait is a *counterclockwise center*.

(9) [8] Suppose you know that a 2×2 matrix $\mathbf{A}$ can be diagonalized as $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$ where

$$\mathbf{V} = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 6 & 0 \\ 0 & -4 \end{pmatrix}.$$

Use this information to compute $e^{t\mathbf{A}}$.

Solution. Because $e^{t\mathbf{A}} = \mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1}$ with $e^{t\mathbf{D}} = \begin{pmatrix} e^{6t} & 0 \\ 0 & e^{-4t} \end{pmatrix}$ and $\mathbf{V}^{-1} = \frac{1}{5}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$,

$$\begin{aligned} e^{t\mathbf{A}} = \mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1} &= \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} e^{6t} & 0 \\ 0 & e^{-4t} \end{pmatrix} \frac{1}{5} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2e^{6t} & e^{6t} \\ -e^{-4t} & 2e^{-4t} \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 4e^{6t} + e^{-4t} & 2e^{6t} - 2e^{-4t} \\ 2e^{6t} - 2e^{-4t} & e^{6t} + 4e^{-4t} \end{pmatrix}. \end{aligned}$$

Alternative Solution. Because

$$\begin{aligned} \mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1} &= \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 6 & 0 \\ 0 & -4 \end{pmatrix} \frac{1}{5} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 12 & 6 \\ -4 & 8 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 20 & 20 \\ 20 & -10 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & -2 \end{pmatrix}, \end{aligned}$$

and because the eigenvalues of $\mathbf{A}$ are 1 ± 5, we obtain

$$\begin{aligned} e^{t\mathbf{A}} &= e^t \left[\mathbf{I}\cosh(5t) + (\mathbf{A} - \mathbf{I})\frac{\sinh(5t)}{5} \right] = e^t \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cosh(5t) + \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \frac{\sinh(5t)}{5} \right] \\ &= e^t \begin{pmatrix} \cosh(5t) + \frac{3}{5}\sinh(5t) & \frac{4}{5}\sinh(5t) \\ \frac{4}{5}\sinh(5t) & \cosh(5t) - \frac{3}{5}\sinh(5t) \end{pmatrix}. \end{aligned}$$

This is equivalent to the solution given previously.