Sample Problems for First In-Class Exam
Math 246, Spring 2009, Professor David Levermore

(1) (a) Write a MATLAB command that evaluates the definite integral

$$\int_0^\infty \frac{r}{1+r^4}\,\mathrm{d}r\,.$$

(b) Sketch the graph that you expect would be produced by the following MATLAB commands.

```
[x, y] = meshgrid(−5:0.5:5,−5:0.2:5)
contour(x, y, x.ˆ2 + y.ˆ2, [25, 25])
axis square
```

(2) Find the explicit solution for each of the following initial-value problems and identify its interval of existence (definition).

(a) $\dfrac{\mathrm{d}z}{\mathrm{d}t} = \dfrac{\cos(t) - z}{1+t}\,, \quad z(0) = 2.$

(b) $\dfrac{\mathrm{d}u}{\mathrm{d}z} = e^u + 1\,, \quad u(0) = 0.$

(3) Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 4y^2 - y^4\,.$$

(a) Find all of its stationary (equilibrium) solutions and classify each as being either stable, unstable, or semistable.
(b) If $y(0) = 1$, how does the solution $y(t)$ behave as $t \to \infty$?
(c) If $y(0) = -1$, how does the solution $y(t)$ behave as $t \to \infty$?
(d) Sketch a graph of y versus t showing the direction field and several solution curves. The graph should show all the stationary solutions as well as solution curves above and below each of them. Every value of y should lie on at least one sketched solution curve.

(4) A tank initially contains 100 liters of pure water. Beginning at time $t = 0$ brine (salt water) with a salt concentration of 2 grams per liter (g/l) flows into the tank at a constant rate of 3 liters per minute (l/min) and the well-stirred mixture flows out of the tank at the same rate. Let $S(t)$ denote the mass (g) of salt in the tank at time $t \geq 0$.
(a) Write down an initial-value problem that governs $S(t)$.
(b) Is $S(t)$ an increasing or decreasing function of t? (Give your reasoning.)
(c) What is the behavior of $S(t)$ as $t \to \infty$? (Give your reasoning.)
(d) Derive an explicit formula for $S(t)$.

(5) Suppose you are using the Heun-midpoint method to numerically approximate the solution of an initial-value problem over the time interval $[0,5]$. By what factor would you expect the error to decrease when you increase the number of time steps taken from 500 to 2000.

(6) Give an implicit general solution to each of the following differential equations.
   (a) $\left(\dfrac{y}{x} + 3x\right) \mathrm{d}x + \big(\log(x) - y\big) \mathrm{d}y = 0\,.$
   (b) $(x^2 + y^3 + 2x) \, \mathrm{d}x + 3y^2 \, \mathrm{d}y = 0\,.$

(7) A 2 kilogram (kg) mass initially at rest is dropped in a medium that offers a resistance of $v^2/40$ newtons ($=$ kg m/sec^2) where v is the downward velocity (m/sec) of the mass. The gravitational acceleration is 9.8 m/sec^2.
   (a) What is the terminal velocity of the mass?
   (b) Write down an initial-value problem that governs v as a function of time. (You do not have to solve it!)

(8) Consider the following MATLAB function M-file.

```
function [t,y] = solveit(ti, yi, tf, n)

h = (tf - ti)/n;
t = zeros(n + 1, 1);
y = zeros(n + 1, 1);
t(1) = ti;
y(1) = yi;
for i = 1:n
z = t(i)^4 + y(i)^2;
t(i + 1) = t(i) + h;
y(i + 1) = y(i) + (h/2)*(z + t(i + 1)^4 + (y(i) + h*z)^2);
end
```

   (a) What is the initial-value problem being approximated numerically?
   (b) What is the numerical method being used?
   (c) What are the output values of t(2) and y(2) that you would expect for input values of ti = 1, yi = 1, tf = 5, n = 20?