Solutions of Sample Problems for First In-Class Exam
Math 246, Spring 2009, Professor David Levermore

(1) (a) Write a MATLAB command that evaluates the definite integral

$$\int_0^\infty \frac{r}{1+r^4}\,\mathrm{d}r\,.$$

Solution: The simplest solution is

```
int('x/(1+x^4)','x',0,inf) ,
```

where you can replace x by any other letter our use Inf instead of inf.

(b) Sketch the graph that you expect would be produced by the following MATLAB commands.

```
[x, y] = meshgrid(−5:0.5:5,−5:0.2:5)
contour(x, y, x.^2 + y.^2, [25, 25])
axis square
```

Solution: Your sketch should show both x and y axes marked from -5 to 5 and a single circle of radius 5 centered at the origin. The tick marks on the axes should mark intervals of length .5.

(2) Find the explicit solution for each of the following initial-value problems and identify its interval of existence (definition).

(a) $\dfrac{\mathrm{d}z}{\mathrm{d}t} = \dfrac{\cos(t) - z}{1+t}\,, \quad z(0) = 2.$

Solution: This equation is *linear* in z, so write it in the linear normal form

$$\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{z}{1+t} = \frac{\cos(t)}{1+t}\,.$$

An integrating factor is given by

$$\exp\left(\int_0^t \frac{1}{1+s}\,\mathrm{d}s\right) = \exp\big(\log(1+t)\big) = 1+t\,,$$

Upon multiplying the equation by $(1+t)$, one finds that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big((1+t)z\big) = \cos(t)\,,$$

which is then integrated to obtain

$$(1+t)z = \sin(t) + c\,.$$

The integration constant c is found through the initial condition $z(0) = 2$ by setting $t = 0$ and $z = 0$, whereby

$$c = (1+0)2 - \sin(0) = 2\,.$$

Hence, upon solving explicitly for z, the solution is

$$z = \frac{2+\sin(t)}{1+t}\,.$$

The interval of existence for this solution is $t > -1$.

(b) $\dfrac{\mathrm{d}u}{\mathrm{d}z} = e^u + 1\,, \quad u(0) = 0.$

Solution: This equation is *autonomus* (and therefore *separable*), so write it in the separated differential form

$$\frac{1}{e^u + 1}\,\mathrm{d}u = \mathrm{d}z\,.$$

This equation can be integrated to obtain

$$z = \int \frac{1}{e^u + 1}\,\mathrm{d}u = \int \frac{e^{-u}}{1 + e^{-u}}\,\mathrm{d}u = -\log(1 + e^{-u}) + c\,.$$

The integration constant c is found through the initial condition $u(0) = 0$ by setting $z = 0$ and $u = 0$, whereby

$$c = 0 + \log(1 + e^0) = \log(2)\,.$$

Hence, the solution is given implicitly by

$$z = -\log(1 + e^{-u}) + \log(2) = -\log\left(\frac{1 + e^{-u}}{2}\right).$$

This may be solve for u as follows:

$$\begin{aligned} e^{-z} &= \frac{1 + e^{-u}}{2}\,, \\ 2e^{-z} - 1 &= e^{-u}\,, \\ u &= -\log(2e^{-z} - 1)\,. \end{aligned}$$

The interval of existence for this solution is $z < \log(2)$.

(3) Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 4y^2 - y^4\,.$$

(a) Find all of its stationary (equilibrium) solutions and classify each as being either stable, unstable, or semistable.

Solution: The right-hand side of the equation factors as

$$4y^2 - y^4 = y^2(4 - y^2) = y^2(2 + y)(2 - y)\,,$$

which implies that $y = -2$, $y = 0$, and $y = 2$ are all of its stationary solutions. A sign analysis of $y^2(2 + y)(2 - y)$ then shows that

$$\begin{aligned} &\frac{\mathrm{d}y}{\mathrm{d}t} > 0 \qquad \text{when } -2 < y < 0 \text{ or } 0 < y < 2\,, \\ &\frac{\mathrm{d}y}{\mathrm{d}t} < 0 \qquad \text{when } -\infty < y < -2 \text{ or } 2 < y < \infty\,. \end{aligned}$$

The phase-line for this equation is therefore

$$\begin{array}{ccccccc} - & & + & & + & & - \\ \longleftarrow\longleftarrow\longleftarrow\longleftarrow & \bullet & \longrightarrow\longrightarrow\longrightarrow\longrightarrow & \bullet & \longrightarrow\longrightarrow\longrightarrow\longrightarrow & \bullet & \longleftarrow\longleftarrow\longleftarrow\longleftarrow \\ & -2 & & 0 & & 2 & \\ & \text{unstable} & & \text{semistable} & & \text{stable} & \end{array}$$

(b) If $y(0) = 1$, how does the solution $y(t)$ behave as $t \to \infty$?

Solution: It is clear from the answer to (a) that

$$\frac{\mathrm{d}y}{\mathrm{d}t} > 0 \quad \text{when } 0 < y < 2\,,$$

so that $y(t) \to 2$ as $t \to \infty$ if $y(0) = 1$.

(c) If $y(0) = -1$, how does the solution $y(t)$ behave as $t \to \infty$?

Solution: It is clear from the answer to (a) that

$$\frac{\mathrm{d}y}{\mathrm{d}t} > 0 \quad \text{when } -2 < y < 0\,,$$

so that $y(t) \to 0$ as $t \to \infty$ if $y(0) = -1$.

(d) Sketch a graph of y versus t showing the direction field and several solution curves. The graph should show all the stationary solutions as well as solution curves above and below each of them. Every value of y should lie on at least one sketched solution curve.

Solution: Will be given during the review session.

(4) A tank initially contains 100 liters of pure water. Beginning at time $t = 0$ brine (salt water) with a salt concentration of 2 grams per liter (g/l) flows into the tank at a constant rate of 3 liters per minute (l/min) and the well-stirred mixture flows out of the tank at the same rate. Let $S(t)$ denote the mass (g) of salt in the tank at time $t \geq 0$.

(a) Write down an initial-value problem that governs $S(t)$.

Solution: Because water flows in and out of the tank at the same rate, the tank will contain 100 liters of salt water for every $t > 0$. The salt concentration of the water in the tank at time t will therefore be $S(t)/100$ g/l. Because this is also the concentration of the outflow, $S(t)$, the mass of salt in the tank at time t, will satisfy

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \text{RATE IN} - \text{RATE OUT} = 2 \cdot 3 - \frac{S}{100} \cdot 3 = 6 - \tfrac{3}{100} S\,.$$

Because there is no salt in the tank initially, the initial-value problem that governs $S(t)$ is

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 6 - \tfrac{3}{100} S\,, \qquad S(0) = 0\,.$$

(b) Is $S(t)$ an increasing or decreasing function of t? (Give your reasoning.)

Solution: One sees from part (a) that

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \tfrac{3}{100}(200 - S) > 0 \quad \text{for } S < 200\,,$$

whereby $S(t)$ is an increasing function of t that will approach the stationary value of 200 g as $t \to \infty$.

(c) What is the behavior of $S(t)$ as $t \to \infty$? (Give your reasoning.)

Solution: The argument given for part (b) already shows that $S(t)$ is an increasing function of t that approaches the stationary value of 200 g as $t \to \infty$.

(d) Derive an explicit formula for $S(t)$.

Solution: The differential equation given in the answer to part (a) is linear, so write it in the form

$$\frac{\mathrm{d}S}{\mathrm{d}t} + \tfrac{3}{100} S = 6\,.$$

An integrating factor is $e^{\frac{3}{100}t}$, whereby

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(e^{\frac{3}{100}t} S\big) = 6 e^{\frac{3}{100}t}\,.$$

This is the integrated to obtain

$$e^{\frac{3}{100}t} S = 200 e^{\frac{3}{100}t} + c\,.$$

The integration constant c is found by setting $t = 0$ and $S = 0$, whereby

$$c = e^0 \cdot 0 - 200 \cdot e^0 = -200\,.$$

Then solving for S gives

$$S(t) = 200 - 200 e^{-\frac{3}{100}t}\,.$$

(5) Suppose you are using the Heun-midpoint method to numerically approximate the solution of an initial-value problem over the time interval $[0, 5]$. By what factor would you expect the error to decrease when you increase the number of time steps taken from 500 to 2000.

Solution: The Heun-midpoint method is second order, which means its (global) error scales like h^2 where h is the time step. When the number of time steps taken increases from 500 to 2000, the time step h decreases by a factor of 4. The error will therefore decrease (like h^2) by a factor of $4^2 = 16$.

(6) Give an implicit general solution to each of the following differential equations.

(a) $\left(\dfrac{y}{x} + 3x\right) \mathrm{d}x + \big(\log(x) - y\big)\, \mathrm{d}y = 0\,.$

Solution: Because

$$\partial_y\left(\frac{y}{x} + 3x\right) = \frac{1}{x} \qquad = \qquad \partial_x\big(\log(x) - y\big) = \frac{1}{x}\,,$$

the equation is *exact*. You can therefore find $H(x, y)$ such that

$$\partial_x H(x, y) = \frac{y}{x} + 3x\,, \qquad \partial_y H(x, y) = \log(x) - y\,.$$

The first of these equations implies that

$$H(x, y) = y \log(x) + \tfrac{3}{2} x^2 + h(y)\,.$$

Plugging this into the second equation then shows that

$$\log(x) - y = \partial_y H(x, y) = \log(x) + h'(y)\,.$$

Hence, $h'(y) = -y$, which yields $h(y) = -\frac{1}{2} y^2$. The general solution is therefore governed implicitly by

$$y \log(x) + \tfrac{3}{2} x^2 - \tfrac{1}{2} y^2 = c\,,$$

where c is an arbitrary constant.

(b) $(x^2 + y^3 + 2x)\,\mathrm{d}x + 3y^2\,\mathrm{d}y = 0\,.$

Solution: Because

$$\partial_y(x^2 + y^3 + 2x) = 3y^2 \qquad \neq \qquad \partial_x(3y^2) = 0\,,$$

the equation is *not exact*. Seek an integrating factor $\mu(x, y)$ such that

$$\partial_y\big((x^2 + y^3 + 2x)\mu\big) = \partial_x(3y^2\mu)\,.$$

This means that μ must satisfy

$$(x^2 + y^3 + 2x)\partial_y\mu + 3y^2\mu = 3y^2\partial_x\mu\,.$$

If you assume that μ depends only on x (so that $\partial_y\mu = 0$) then this reduces to

$$\mu = \partial_x\mu\,,$$

which depends only on x. One sees from this that $\mu = e^x$ is an integrating factor. This implies that

$$(x^2 + y^3 + 2x)e^x\,\mathrm{d}x + 3y^2e^x\,\mathrm{d}y = 0 \quad \text{is exact}\,.$$

You can therefore find $H(x, y)$ such that

$$\partial_x H(x, y) = (x^2 + y^3 + 2x)e^x\,, \qquad \partial_y H(x, y) = 3y^2e^x\,.$$

The second of these equations implies that

$$H(x, y) = y^3e^x + h(x)\,.$$

Plugging this into the first equation then yields

$$(x^2 + y^3 + 2x)e^x = \partial_x H(x, y) = y^3e^x + h'(x)\,.$$

Hence, h satisfies

$$h'(x) = (x^2 + 2x)e^x\,.$$

This can be integrated to obtain $h(x) = x^2e^x$. The general solution is therefore governed implicitly by

$$(y^3 + x^2)e^x = c\,,$$

where c is an arbitrary constant.

(7) A 2 kilogram (kg) mass initially at rest is dropped in a medium that offers a resistance of $v^2/40$ newtons (= kg m/sec^2) where v is the downward velocity (m/sec) of the mass. The gravitational acceleration is 9.8 m/sec^2.

(a) What is the terminal velocity of the mass?

Solution: The terminal velocity is the velocity at which the force of resistence balances that of gravity. This happens when

$$\tfrac{1}{40}v^2 = mg = 2 \cdot 9.8\,.$$

Upon solving this for v one obtains

$$\begin{aligned} v &= \sqrt{40 \cdot 2 \cdot 9.8}\ \text{m/sec} \qquad \text{(full marks)} \\ &= \sqrt{4 \cdot 2 \cdot 98} = \sqrt{4 \cdot 2 \cdot 2 \cdot 49} \\ &= \sqrt{4^2 \cdot 7^2} = 4 \cdot 7 = 28\ \text{m/sec}\,. \end{aligned}$$

(b) Write down an initial-value problem that governs v as a function of time. (You do not have to solve it!)

Solution: The net downward force on the falling mass is the force of gravity minus the force of resistence. By Newton ($ma = F$), this leads to

$$m\frac{dv}{dt} = mg - \tfrac{1}{40}v^2 .$$

Because $m = 2$ and $g = 9.8$, and because the mass is initially at rest, this yields the initial-value problem

$$\frac{dv}{dt} = 9.8 - \tfrac{1}{80}v^2 , \qquad v(0) = 0 .$$

(8) Consider the following MATLAB function M-file.

```
function [t,y] = solveit(ti, yi, tf, n)

h = (tf - ti)/n;
t = zeros(n + 1, 1);
y = zeros(n + 1, 1);
t(1) = ti;
y(1) = yi;
for i = 1:n
z = t(i)^4 + y(i)^2;
t(i + 1) = t(i) + h;
y(i + 1) = y(i) + (h/2)*(z + t(i + 1)^4 + (y(i) + h*z)^2);
end
```

(a) What is the initial-value problem being approximated numerically?

Solution: The initial-value problem being approximated is

$$\frac{dy}{dt} = t^4 + y^2 , \qquad y(t_o) = y_o .$$

(b) What is the numerical method being used?

Solution: The Heun-Trapezoidal (improved Euler) method is being used.

(c) What are the output values of t(2) and y(2) that you would expect for input values of ti = 1, yi = 1, tf = 5, n = 20?

Solution: The time step is given by h = (tf - ti)/n = (5 - 1)/20 = 1/5 = .2. The initial time and data are given by t(1) = ti = 1 and y(1) = yi = 1. One then has

$$\begin{aligned}
\mathrm{t}(2) &= \mathrm{t}(1) + \mathrm{h} = 1 + .2 = 1.2 , \\
\mathrm{z} &= \mathrm{t}(1)^4 + \mathrm{y}(1)^2 = 1 + 1 = 2 , \\
\mathrm{y}(2) &= \mathrm{y}(1) + (\mathrm{h}/2)\,\big(\mathrm{z} + \mathrm{t}(2)^4 + (\mathrm{y}(1) + \mathrm{h\,z})^2\big) \\
&= 1 + .1\big(2 + (1.2)^4 + (1 + .2\cdot 2)^2\big) .
\end{aligned}$$