Sample Problems for Second In-Class Exam
Math 246, Spring 2009, Professor David Levermore

(1) Give the interval of existence for the solution of the initial-value problem

$$\frac{\mathrm{d}^3x}{\mathrm{d}t^3} + \frac{\cos(3t)}{4-t}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{e^{-2t}}{1+t}, \qquad x(2) = x'(2) = x''(2) = 0\,.$$

(2) Let L be a linear ordinary differential operator with constant coefficients. Suppose that all the roots of its characteristic polynomial (listed with their multiplicities) are $-2+i3$, $-2-i3$, $i7$, $i7$, $-i7$, $-i7$, 5, 5, 5, -3, 0, 0.
(a) Give the order of L.
(b) Give a general real solution of the homogeneous equation $\mathrm{L}y = 0$.

(3) Let $\mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}$. Solve each of the following initial-value problems.
(a) $\mathrm{D}^2y + 4\mathrm{D}y + 4y = 0\,, \qquad y(0) = 1\,, \quad y'(0) = 0\,.$
(b) $\mathrm{D}^2y + 9y = 20e^t\,, \qquad y(0) = 0\,, \quad y'(0) = 0\,.$

(4) Let $\mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}$. Give a general real solution for each of the following equations.
(a) $\mathrm{D}^2y + 4\mathrm{D}y + 5y = 3\cos(2t)\,.$
(b) $\mathrm{D}^2y - y = t\,e^t\,.$
(c) $\mathrm{D}^2y - y = \dfrac{1}{1+e^t}\,.$

(5) Let $\mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}$. Consider the equation

$$\mathrm{L}y = \mathrm{D}^2y - 6\mathrm{D}y + 25y = e^{t^2}\,.$$

(a) Compute the Green function $g(t)$ associated with L.
(b) Use the Green function to express a particular solution $Y_P(t)$ in terms of definite integrals.

(6) The functions t and t^2 are solutions of the homogeneous equation

$$t^2\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2t\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = 0 \qquad \text{over } t > 0\,.$$

(You do not have to check that this is true!)
(a) Compute their Wronskian.
(b) Solve the initial-value problem

$$t^2\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2t\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = t^3e^t\,, \qquad y(1) = y'(1) = 0\,, \qquad \text{over } t > 0\,.$$

Try to evaluate all definite integrals explicitly.

(7) What answer will be produced by the following MATLAB commands?

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>> ode1 = 'D2y + 2*Dy + 5*y = 16*exp(t)';
>> dsolve(ode1, 't')
ans =
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(8) The vertical displacement of a mass on a spring is given by

$$h(t) = 4e^{-t}\cos(7t) - 3e^{-t}\sin(7t)\,.$$

(a) Express $h(t)$ in the form $h(t) = Ae^{-t}\cos(\omega t - \delta)$ with $A > 0$ and $0 \leq \delta < 2\pi$, identifying the quasiperiod and phase of the oscillation. (The phase may be expressed in terms of an inverse trig function.)
(b) Sketch the solution over $0 \leq t \leq 2$.

(9) When a mass of 4 grams is hung vertically from a spring, at rest it stretches the spring 9.8 cm. (Gravitational acceleration is $g = 980$ cm/sec^2.) At $t = 0$ the mass is displaced 3 cm above its equilibrium position and is released with no initial velocity. It moves in a medium that imparts a drag force of 2 dynes (1 dyne = 1 gram cm/sec^2) when the speed of the mass is 4 cm/sec. There are no other forces. (Assume that the spring force is proportional to displacement and that the drag force is proportional to velocity.)
(a) Formulate an initial-value problem that governs the motion of the mass for $t > 0$. (DO NOT solve this initial-value problem, just write it down!)
(b) What is the natural frequency of the spring?
(c) Show that the system is under damped and find its quasifrequency.

(10) Compute the Laplace transform of $f(t) = t\,e^{3t}$ from its definition.

(11) Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 13y = f(t)\,, \qquad y(0) = 4\,, \quad y'(0) = 1\,,$$

where

$$f(t) = \begin{cases} \cos(t) & \text{for } 0 \leq t < 2\pi\,, \\ t - 2\pi & \text{for } t \geq 2\pi\,. \end{cases}$$

You may refer to the table on the last page. DO NOT take the inverse Laplace transform to find $y(t)$, just solve for $Y(s)$!

(12) Find the inverse Laplace transforms of the following functions. You may refer to the table on the last page.

(a) $F(s) = \dfrac{2}{(s+5)^2}\,,$

(b) $F(s) = \dfrac{3s}{s^2 - s - 6}\,,$

(c) $F(s) = \dfrac{(s-2)e^{-3s}}{s^2 - 4s + 5}\,.$

A Short Table of Laplace Transforms

$$\mathcal{L}[e^{at}t^n](s) = \frac{n!}{(s-a)^{n+1}} \qquad \text{for } s > a\,,$$

$$\mathcal{L}[e^{at}\cos(bt)](s) = \frac{s-a}{(s-a)^2+b^2} \qquad \text{for } s > a\,,$$

$$\mathcal{L}[e^{at}\sin(bt)](s) = \frac{b}{(s-a)^2+b^2} \qquad \text{for } s > a\,,$$

$$\mathcal{L}[e^{at}f(t)](s) = F(s-a) \qquad \text{where } F(s) = \mathcal{L}[f(t)](s)\,,$$

$$\mathcal{L}[t^n f(t)](s) = (-1)^n F^{(n)}(s) \qquad \text{where } F(s) = \mathcal{L}[f(t)](s)\,,$$

$$\mathcal{L}[u(t-c)f(t-c)](s) = e^{-cs}F(s) \qquad \text{where } F(s) = \mathcal{L}[f(t)](s)$$

and u is the step function.