Solutions to Sample Problems for Third In-Class Exam
Math 246, Spring 2009, Professor David Levermore

(1) Consider the matrices

$$\mathbf{A} = \begin{pmatrix} -i2 & 1+i \\ 2+i & -4 \end{pmatrix} , \qquad \mathbf{B} = \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix} .$$

Compute the matrices

(a) $\mathbf{A}^T$,

Solution. The transpose of $\mathbf{A}$ is

$$\mathbf{A}^T = \begin{pmatrix} -i2 & 2+i \\ 1+i & -4 \end{pmatrix} .$$

(b) $\overline{\mathbf{A}}$,

Solution. The conjugate of $\mathbf{A}$ is

$$\overline{\mathbf{A}} = \begin{pmatrix} i2 & 1-i \\ 2-i & -4 \end{pmatrix} .$$

(c) $\mathbf{A}^*$,

Solution. The adjoint of $\mathbf{A}$ is

$$\mathbf{A}^* = \begin{pmatrix} i2 & 2-i \\ 1-i & -4 \end{pmatrix} .$$

(d) $5\mathbf{A} - \mathbf{B}$,

Solution. The difference of $5\mathbf{A}$ and $\mathbf{B}$ is given by

$$5\mathbf{A} - \mathbf{B} = \begin{pmatrix} -i10 & 5+i5 \\ 10+i5 & -20 \end{pmatrix} - \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix} = \begin{pmatrix} -7-i10 & -1+i5 \\ 2+i5 & -27 \end{pmatrix} .$$

(e) $\mathbf{AB}$,

Solution. The product of $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} -i2 & 1+i \\ 2+i & -4 \end{pmatrix} \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix} \\ &= \begin{pmatrix} -i2\cdot 7 + (1+i)\cdot 8 & -i2\cdot 6 + (1+i)\cdot 7 \\ (2+i)\cdot 7 - 4\cdot 8 & (2+i)\cdot 6 - 4\cdot 7 \end{pmatrix} \\ &= \begin{pmatrix} 8-i6 & 7-i5 \\ -18+i7 & -16+i6 \end{pmatrix} . \end{aligned}$$

(f) $\mathbf{B}^{-1}$.

Solution. Observe that it is clear that $\mathbf{B}$ has an inverse because

$$\det(\mathbf{B}) = \det\begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix} = 7 \cdot 7 - 6 \cdot 8 = 49 - 48 = 1\,.$$

The inverse of $\mathbf{B}$ is given by

$$\mathbf{B}^{-1} = \frac{1}{\det(\mathbf{B})}\begin{pmatrix} 7 & -6 \\ -8 & 7 \end{pmatrix} = \begin{pmatrix} 7 & -6 \\ -8 & 7 \end{pmatrix}\,.$$

(2) Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 3 \\ 4 & -1 \end{pmatrix}\,.$$

(a) Find all the eigenvalues of $\mathbf{A}$.

Solution. The characteristic polynomial of $\mathbf{A}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z - 15 = (z-1)^2 - 16\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are 1 ± 4, or simply -3 and 5.

(b) For each eigenvalue of $\mathbf{A}$ find all of its eigenvectors.

Solution (using the Cayley-Hamilton method from notes). One has

$$\mathbf{A} + 3\mathbf{I} = \begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix}\,, \qquad \mathbf{A} - 5\mathbf{I} = \begin{pmatrix} -2 & 3 \\ 4 & -6 \end{pmatrix}\,.$$

Every nonzero column of $\mathbf{A} - 5\mathbf{I}$ has the form

$$\alpha_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{for some } \alpha_1 \neq 0\,.$$

These are all the eigenvectors associated with -3. Similarly, every nonzero column of $\mathbf{A} + 3\mathbf{I}$ has the form

$$\alpha_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \text{for some } \alpha_2 \neq 0\,.$$

These are all the eigenvectors associated with 5.

(c) Diagonalize $\mathbf{A}$.

Solution. If you use the eigenpairs

$$\left(-3\,, \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right)\,, \qquad \left(5\,, \begin{pmatrix} 3 \\ 2 \end{pmatrix}\right)\,,$$

then set

$$\mathbf{V} = \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix}\,, \qquad \mathbf{D} = \begin{pmatrix} -3 & 0 \\ 0 & 5 \end{pmatrix}\,.$$

Because $\det(\mathbf{V}) = 1 \cdot 2 - (-2) \cdot 3 = 2 + 6 = 8$, you see that

$$\mathbf{V}^{-1} = \frac{1}{8}\begin{pmatrix} 2 & -3 \\ 2 & 1 \end{pmatrix}\,.$$

You conclude that $\mathbf{A}$ has the diagonalization

$$\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1} = \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} -3 & 0 \\ 0 & 5 \end{pmatrix} \frac{1}{8} \begin{pmatrix} 2 & -3 \\ 2 & 1 \end{pmatrix} .$$

You do not have to multiply these matrices out for full credit. You can do so to check your answer. (You should get $\mathbf{A}$ if you do.) Had you started with different eigenpairs, the steps would be the same as above but you would obtain a different diagonalization.

(3) Solve each of the following initial-value problems.

(a) $$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} , \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} .$$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - z - 12 = (z+3)(z-4) .$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are -3 and 4. These have the form $\frac{1}{2} \pm \frac{7}{2}$. One therefore has

$$\begin{aligned} e^{t\mathbf{A}} &= e^{\frac{1}{2}t} \left[\mathbf{I} \cosh\left(\tfrac{7}{2}t\right) + \left(\mathbf{A} - \tfrac{1}{2}\mathbf{I}\right) \frac{\sinh\left(\frac{7}{2}t\right)}{\frac{7}{2}} \right] \\ &= e^{\frac{1}{2}t} \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cosh\left(\tfrac{7}{2}t\right) + \begin{pmatrix} \frac{3}{2} & 2 \\ 5 & -\frac{3}{2} \end{pmatrix} \frac{\sinh\left(\frac{7}{2}t\right)}{\frac{7}{2}} \right] \\ &= e^{\frac{1}{2}t} \begin{pmatrix} \cosh\left(\frac{7}{2}t\right) + \frac{3}{7}\sinh\left(\frac{7}{2}t\right) & \frac{4}{7}\sinh\left(\frac{7}{2}t\right) \\ \frac{10}{7}\sinh\left(\frac{7}{2}t\right) & \cosh\left(\frac{7}{2}t\right) - \frac{3}{7}\sinh\left(\frac{7}{2}t\right) \end{pmatrix} . \end{aligned}$$

The solution of the initial-value problem is therefore

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = e^{t\mathbf{A}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= e^{\frac{1}{2}t} \begin{pmatrix} \cosh\left(\frac{7}{2}t\right) + \frac{3}{7}\sinh\left(\frac{7}{2}t\right) & \frac{4}{7}\sinh\left(\frac{7}{2}t\right) \\ \frac{10}{7}\sinh\left(\frac{7}{2}t\right) & \cosh\left(\frac{7}{2}t\right) - \frac{3}{7}\sinh\left(\frac{7}{2}t\right) \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= e^{\frac{1}{2}t} \begin{pmatrix} \cosh\left(\frac{7}{2}t\right) - \frac{1}{7}\sinh\left(\frac{7}{2}t\right) \\ -\cosh\left(\frac{7}{2}t\right) + \frac{13}{7}\sinh\left(\frac{7}{2}t\right) \end{pmatrix} . \end{aligned}$$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z + 5 = (z-1)^2 + 4\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $1 \pm i2$. One therefore has

$$\begin{aligned} e^{t\mathbf{A}} &= e^t\left[\mathbf{I}\cos(2t) + \big(\mathbf{A} - \mathbf{I}\big)\frac{\sin(2t)}{2}\right] \\ &= e^t\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(2t) + \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\frac{\sin(2t)}{2}\right] \\ &= e^t\begin{pmatrix} \cos(2t) & \frac{1}{2}\sin(2t) \\ -2\sin(2t) & \cos(2t) \end{pmatrix}. \end{aligned}$$

The solution of the initial-value problem is therefore

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}}\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = e^{t\mathbf{A}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= e^t\begin{pmatrix} \cos(2t) & \frac{1}{2}\sin(2t) \\ -2\sin(2t) & \cos(2t) \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= e^t\begin{pmatrix} \cos(2t) + \frac{1}{2}\sin(2t) \\ -2\sin(2t) + \cos(2t) \end{pmatrix}. \end{aligned}$$

(4) Compute $e^{t\mathbf{A}}$ for $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$.

Solution. The characteristic polynomial of $\mathbf{A}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z - 3 = (z-1)^2 - 4\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are 1 ± 2. One then has

$$\begin{aligned} e^{t\mathbf{A}} &= e^t\left[\mathbf{I}\cosh(2t) + \big(\mathbf{A} - \mathbf{I}\big)\frac{\sinh(2t)}{2}\right] \\ &= e^t\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cosh(2t) + \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}\frac{\sinh(2t)}{2}\right] \\ &= e^t\begin{pmatrix} \cosh(2t) & 2\sinh(2t) \\ \frac{1}{2}\sinh(2t) & \cosh(2t) \end{pmatrix}. \end{aligned}$$

(5) Find a general solution for each of the following systems.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z + 1 = (z-1)^2 .$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which is 1, a double root. One therefore has

$$\begin{aligned} e^{t\mathbf{A}} &= e^t\big[\mathbf{I} + (\mathbf{A} - \mathbf{I})t\big] = e^t\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} t\right] \\ &= e^t\begin{pmatrix} 1+2t & -4t \\ t & 1-2t \end{pmatrix} . \end{aligned}$$

A general solution is therefore given by

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = e^t\begin{pmatrix} 1+2t & -4t \\ t & 1-2t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= e^t\begin{pmatrix} c_1(1+2t) - 4c_2t \\ c_1t + c_2(1-2t) \end{pmatrix} . \end{aligned}$$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 16 .$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $\pm i4$. One therefore has

$$\begin{aligned} e^{t\mathbf{A}} &= \left[\mathbf{I}\cos(4t) + \mathbf{A}\frac{\sin(4t)}{4}\right] = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(4t) + \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}\frac{\sin(4t)}{4}\right] \\ &= \begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) & -\frac{5}{4}\sin(4t) \\ \sin(4t) & \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix} . \end{aligned}$$

A general solution is therefore given by

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) & -\frac{5}{4}\sin(4t) \\ \sin(4t) & \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= \begin{pmatrix} c_1\big(\cos(4t) + \frac{1}{2}\sin(4t)\big) - c_2\frac{5}{4}\sin(4t) \\ c_1\sin(4t) + c_2\big(\cos(4t) - \frac{1}{2}\sin(4t)\big) \end{pmatrix} . \end{aligned}$$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 5 & 4 \\ -5 & 1 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 6z + 25 = (z-3)^2 + 16\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $3 \pm i4$. One therefore has

$$\begin{aligned} e^{t\mathbf{A}} &= e^{3t}\left[\mathbf{I}\cos(4t) + (\mathbf{A} - 3\mathbf{I})\frac{\sin(4t)}{4}\right] \\ &= e^{3t}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(4t) + \begin{pmatrix} 2 & 4 \\ -5 & -2 \end{pmatrix}\frac{\sin(4t)}{4}\right] \\ &= e^{3t}\begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) & \sin(4t) \\ -\frac{5}{4}\sin(4t) & \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix}. \end{aligned}$$

A general solution is therefore given by

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = e^{3t}\begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) & \sin(4t) \\ -\frac{5}{4}\sin(4t) & \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= e^{3t}\begin{pmatrix} c_1\big(\cos(4t) + \frac{1}{2}\sin(4t)\big) + c_2\sin(4t) \\ -c_1\frac{5}{4}\sin(4t) + c_2\big(\cos(4t) - \frac{1}{2}\sin(4t)\big) \end{pmatrix}. \end{aligned}$$

(6) Sketch the phase-plane portrait for each of the systems in the previous problem. Indicate typical trajectories. For each portrait identify its type and give a reason why the origin is either attracting, stable, unstable, or repelling.

(a) **Solution.** Because the characteristic polynomial of $\mathbf{A}$ is $p(z) = (z-1)^2$, one sees that $\mu = 1$ and $\delta = 0$. Because

$$\mathbf{A} - \mathbf{I} = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix},$$

we see that the eigenvectors associated with 1 are

$$\alpha\begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{for some } \alpha \neq 0\,.$$

Because $\mu = 1 > 0$, $\delta = 0$, and $a_{21} > 0$ the phase portrait is a *counterclockwise twist source*. The origin is thereby *repelling*. The phase portrait should show there is one trajectory that emerges from the origin on each side of the line $y = x/2$. Every other trajectory emerges from the origin with a clockwise twist.

(b) **Solution.** Because the characteristic polynomial of $\mathbf{A}$ is $p(z) = z^2 + 16$, one sees that $\mu = 0$ and $\delta = -16$. There are no real eigenpairs. Because $\mu = 0$, $\delta = -16 < 0$, and $a_{21} > 0$ the phase portrait is a *counterclockwise center*. The origin is thereby *stable*. The phase portrait should indicate a family of counterclockwise elliptical trajectories that go around the origin.

(c) **Solution.** Because the characteristic polynomial of $\mathbf{A}$ is $p(z) = (z-3)^2 + 16$, one sees that $\mu = 3$ and $\delta = -16$. There are no real eigenpairs. Because $\mu = 3$, $\delta = -16 < 0$, and $a_{21} < 0$ the phase portrait is a *clockwise spiral source.* The origin is thereby *repelling.* The phase portrait should indicate a family of clockwise spiral trajectories that emerge from the origin.

(7) Transform the equation $\dfrac{\mathrm{d}^3u}{\mathrm{d}t^3} + t^2\dfrac{\mathrm{d}u}{\mathrm{d}t} - 3u = \sinh(2t)$ into a first-order system of ordinary differential equations.

Solution: Because the equation is third order, the first order system must have dimension three. The simplest such first order system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \sinh(2t) + 3x_1 - t^2 x_2 \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} u \\ u' \\ u'' \end{pmatrix}.$$

(8) Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} t^4 + 3 \\ 2t^2 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t^2 \\ 3 \end{pmatrix}$.

(a) Compute the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2](t)$.

Solution.

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \det\begin{pmatrix} t^4 + 3 & t^2 \\ 2t^2 & 3 \end{pmatrix} = 3t^4 + 9 - 2t^4 = t^4 + 9\,.$$

(b) Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x}$ wherever $W[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$.

Solution. Let $\mathbf{\Psi}(t) = \begin{pmatrix} t^4 + 3 & t^2 \\ 2t^2 & 3 \end{pmatrix}$. Because $\dfrac{\mathbf{\Psi}(t)}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{\Psi}(t)$, one has

$$\begin{aligned} \mathbf{A}(t) = \frac{\mathbf{\Psi}(t)}{\mathrm{d}t}\mathbf{\Psi}(t)^{-1} &= \begin{pmatrix} 4t^3 & 2t \\ 4t & 0 \end{pmatrix}\begin{pmatrix} t^4 + 3 & t^2 \\ 2t^2 & 3 \end{pmatrix}^{-1} \\ &= \frac{1}{t^4 + 9}\begin{pmatrix} 4t^3 & 2t \\ 4t & 0 \end{pmatrix}\begin{pmatrix} 3 & -t^2 \\ -2t^2 & t^4 + 3 \end{pmatrix} = \frac{1}{t^4 + 9}\begin{pmatrix} 8t^3 & -2t^5 + 6t \\ 12t & -4t^3 \end{pmatrix}. \end{aligned}$$

(c) Give a fundamental matrix $\mathbf{\Psi}(t)$ for the system found in part (b).

Solution. A fundamental matrix is $\mathbf{\Psi}(t) = \begin{pmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) \end{pmatrix} = \begin{pmatrix} t^4 + 3 & t^2 \\ 2t^2 & 3 \end{pmatrix}$.

(d) Compute the Green matrix $\mathbf{G}(t,s)$ for the system found in part (b).

Solution. The Green matrix is given by

$$\begin{aligned} \mathbf{G}(t,s) = \mathbf{\Psi}(t)\mathbf{\Psi}(s)^{-1} &= \begin{pmatrix} t^4 + 3 & t^2 \\ 2t^2 & 3 \end{pmatrix}\frac{1}{s^4 + 9}\begin{pmatrix} 3 & -s^2 \\ -2s^2 & s^4 + 3 \end{pmatrix} \\ &= \frac{1}{s^4 + 9}\begin{pmatrix} 3t^4 + 9 - 2t^2s^2 & t^2s^4 + 3t^2 - t^4s^2 - 3s^2 \\ 6t^2 - 6s^2 & 3s^4 + 9 - 2t^2s^2 \end{pmatrix}. \end{aligned}$$

(e) For the system found in part (b), solve the initial-value problem

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x}\,, \qquad \mathbf{x}(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Solution. The solution of the initial-value problem is given by

$$\begin{aligned} \mathbf{x}(t) = \mathbf{G}(t,1)\mathbf{x}(1) &= \frac{1}{10}\begin{pmatrix} 3t^4+9-2t^2 & 4t^2-t^4-3 \\ 6t^2-6 & 12-2t^2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{10}\begin{pmatrix} 3t^4+9-2t^2 \\ 6t^2-6 \end{pmatrix}. \end{aligned}$$

Alternative Solution. A general solution is given by $\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{c}$ where $\mathbf{\Psi}(t)$ is the fundamental matrix you found in part (c). By imposing the initial condition you find that $\mathbf{x}(1) = \mathbf{\Psi}(1)\mathbf{c}$, whereby $\mathbf{c} = \mathbf{\Psi}(1)^{-1}\mathbf{x}(1)$. The solution of the initial-value problem is therefore given by

$$\begin{aligned} \mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}(1)^{-1}\mathbf{x}(1) &= \begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix}\begin{pmatrix} 1^4+3 & 1^2 \\ 21^2 & 3 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix}\frac{1}{12-2}\begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{10}\begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 3t^4+9-2t^2 \\ 6t^2-6 \end{pmatrix}. \end{aligned}$$

(9) Consider two interconnected tanks filled with brine (salt water). The first tank contains 100 liters and the second contains 50 liters. Brine flows with a concentration of 2 grams of salt per liter flows into the first tank at a rate of 3 liters per hour. Well stirred brine flows from the first tank to the second at a rate of 5 liters per hour, from the second to the first at a rate of 2 liters per hour, and from the second into a drain at a rate of 3 liters per hour. At $t = 0$ there are 5 grams of salt in the first tank and 20 grams in the second. Give an initial-value problem that governs the amount of salt in each tank as a function of time.

Solution: The rates work out so there will always be 100 liters of brine in the first tank and 50 liters in the second. Let $S_1(t)$ be the grams of salt in the first tank and $S_2(t)$ be the grams of salt in the second tank. These are governed by the initial-value problem

$$\begin{aligned} \frac{\mathrm{d}S_1}{\mathrm{d}t} &= 2\cdot 3 + \frac{S_2}{50}2 - \frac{S_1}{100}5\,, \qquad & S_1(0) = 2\,, \\ \frac{\mathrm{d}S_2}{\mathrm{d}t} &= \frac{S_1}{100}5 - \frac{S_2}{50}2 - \frac{S_2}{50}3\,, \qquad & S_2(0) = 20\,. \end{aligned}$$

You could leave the answer in the above form. It can however be simplified to

$$\begin{aligned} \frac{\mathrm{d}S_1}{\mathrm{d}t} &= 6 + \frac{S_2}{25} - \frac{S_1}{20}\,, \qquad & S_1(0) = 2\,, \\ \frac{\mathrm{d}S_2}{\mathrm{d}t} &= \frac{S_1}{20} - \frac{S_2}{10}\,, \qquad & S_2(0) = 20\,. \end{aligned}$$

(10) Given that 1 is an eigenvalue of the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & -1 & 3 \end{pmatrix},$$

find all the eigenvectors of $\mathbf{A}$ associated with 1.

Solution. The eigenvectors of $\mathbf{A}$ associated with 1 are all nonzero vectors $\mathbf{v}$ that satisfy $\mathbf{A}\mathbf{v} = \mathbf{v}$. Equivalently, they are all nonzero vectors $\mathbf{v}$ that satisfy $(\mathbf{A}-\mathbf{I})\mathbf{v} = 0$, which is

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0\,.$$

The entries of $\mathbf{v}$ thereby satisfy the homogeneous linear algebraic system

$$\begin{aligned} v_1 - v_2 + v_3 &= 0\,, \\ v_1 - v_3 &= 0\,, \\ - v_2 + 2v_3 &= 0\,. \end{aligned}$$

You may solve this system either by elimination or by row reduction. For example, by row reduction you obtain

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}.$$

By either method you find that its general solution is

$$v_1 = \alpha\,, \quad v_2 = 2\alpha\,, \quad v_3 = \alpha\,, \qquad \text{for any constant } \alpha\,.$$

The eigenvectors of $\mathbf{A}$ associated with 1 therefore have the form

$$\alpha \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \qquad \text{for any nonzero constant } \alpha\,.$$