Solutions to Sample Final Exam Problems, Math 246, Spring 2009

(1) Consider the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}t} = (9 - y^2)y^2$.

(a) Identify its equilibrium (stationary) points and classify their stability.
(b) Sketch how solutions move in the interval $-5 \le y \le 5$ (its phase-line portrait).
(c) If $y(0) = -1$, how does the solution $y(t)$ behave as $t \to \infty$?

Solution (a,b): The right-hand side factors as $(3 + y)(3 - y)y^2$. The stationary solutions are $y = -3$, $y = 0$, and $y = 3$. A sign analysis of $(3 + y)(3 - y)y^2$ shows that the phase-line portrait for this equation is therefore

$$\begin{array}{ccccccccc} & - & & + & & + & & - & \\ & \leftarrow\leftarrow\leftarrow\leftarrow & \bullet & \rightarrow\rightarrow\rightarrow\rightarrow & \bullet & \rightarrow\rightarrow\rightarrow\rightarrow & \bullet & \leftarrow\leftarrow\leftarrow\leftarrow & y \\ & & -3 & & 0 & & 3 & & \\ & & \text{unstable} & & \text{semistable} & & \text{stable} & & \end{array}$$

Solution (c): The phase-line shows that if $y(0) = -1$ then $y(t) \to 0$ as $t \to \infty$.

(2) Solve (possibly implicitly) each of the following initial-value problems. Identify their intervals of definition.

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}t} + \dfrac{2ty}{1 + t^2} = t^2\,, \quad y(0) = 1\,.$

Solution: This equation is *linear* and is already in normal form. An integrating factor is

$$\exp\left(\int_0^t \frac{2s}{1 + s^2}\,\mathrm{d}s\right) = \exp\big(\log(1 + t^2)\big) = 1 + t^2\,,$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big((1 + t^2)y\big) = (1 + t^2)t^2 = t^2 + t^4\,.$$

Integrate this to obtain

$$(1 + t^2)y = \tfrac{1}{3}t^3 + \tfrac{1}{5}t^5 + c\,.$$

The initial condition $y(0) = 1$ implies that $c = (1 + 0^2)\cdot 1 - \frac{1}{3}0^3 - \frac{1}{5}0^5 = 1$. Therefore

$$y = \frac{1 + \frac{1}{3}t^3 + \frac{1}{5}t^5}{1 + t^2}\,.$$

This solution exists for every t.

(b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + \dfrac{e^x y + 2x}{2y + e^x} = 0\,, \quad y(0) = 0\,.$

Solution: Express this equation in the differential form

$$(e^x y + 2x)\,\mathrm{d}x + (2y + e^x)\,\mathrm{d}y = 0\,.$$

This differential form is *exact* because

$$\partial_y(e^x y + 2x) = e^x \qquad = \qquad \partial_x(2y + e^x) = e^x\,.$$

We can therefore find $H(x, y)$ such that

$$\partial_x H(x, y) = e^x y + 2x\,, \qquad \partial_y H(x, y) = 2y + e^x\,.$$

The first equation implies $H(x,y) = e^x y + x^2 + h(y)$. Plugging this into the second equation gives

$$e^x + h'(y) = 2y + e^x\,,$$

which yields $h'(y) = 2y$. Taking $h(y) = y^2$, the general solution is

$$e^x y + x^2 + y^2 = c\,.$$

The initial condition $y(0) = 0$ implies that $c = e^0 \cdot 0 + 0^2 + 0^2 = 0$. Therefore

$$y^2 + e^x y + x^2 = 0\,.$$

If you had been asked for an explicit solution then the quadratic formula yields

$$y = \frac{-e^x + \sqrt{e^{2x} - 4x^2}}{2}\,.$$

Here the positive square root is taken because that solution satisfies the initial condition. It exists wherever $e^{2x} \geq 4x^2$.

(3) Let $y(t)$ be the solution of the initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y^2 + t^2\,, \qquad y(0) = 1\,.$$

Use two steps of the forward Euler method to approximate $y(0.2)$.

Solution. The forward Euler method is

$$\begin{aligned} f_n &= f(y_n, t_n)\,, \\ y_{n+1} &= y_n + h f_n\,, \\ t_{n+1} &= t_n + h\,, \end{aligned}$$

where h is the time step, t_0 is the initial time, and y_0 is the initial data.

When the forward Euler method is applied with $h = 0.1$, $t_0 = 0$, $y_0 = 1$, and $f(y,t) = y^2 + t^2$ for two steps

$$\begin{aligned} f_0 &= f(y_0, t_0) = y_0^2 + t_0^2 = 1^2 + 0^2 = 1\,, \\ y_1 &= y_0 + h f_0 = 1 + 0.1 \cdot 1 = 1.1\,, \\ t_1 &= t_0 + h = 0 + 0.1 = 0.1\,, \\ f_1 &= f(y_1, t_1) = y_1^2 + t_1^2 = (1.1)^2 + (0.1)^2\,, \\ y_2 &= y_1 + h f_1 = 1.1 + 0.1 \cdot \big((1.1)^2 + (0.1)^2\big)\,. \end{aligned}$$

The approximation is therefore

$$y(0.2) \approx 1.1 + 0.1 \cdot \big((1.1)^2 + (0.1)^2\big)\,.$$

You DO NOT have to work out the arithmetic! If you did then $y_2 = 1.222$.

Remark. You should be able to answer similar questions that ask you to use the imploved Euler (trapeziodal Heun) method.

(4) Give an explicit real-valued general solution of the following equations.

(a) $\dfrac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2\dfrac{\mathrm{d}y}{\mathrm{d}t} + 5y = te^t + \cos(2t)$

Solution. This is a constant coefficient, inhomogeneous, linear equation. Its characteristic polynomial is

$$p(z) = z^2 - 2z + 5 = (z-1)^2 + 4 = (z-1)^2 + 2^2 .$$

This has the conjugate pair of roots $1 \pm i2$, which yields a general solution of the associated homogeneous problem

$$y_H(t) = c_1 e^t \cos(2t) + c_2 e^t \sin(2t) .$$

A particular solution $y_P(t)$ can be found by the method of undetermined coefficients. The characteristics of the forcing terms te^t and $\cos(2t)$ are $r + is = 1$ and $r + is = i2$ respectively. Because these characteristics are different, they should be treated separately. This can be done using either KEY identity evaluation or direct substitution.

KEY Indentity Evaluations. The forcing term $t\,e^t$ has degree $d = 1$ and characteristic $r + is = 1$, which is a root of $p(z)$ of multiplicity $m = 0$. Because $m + d = 1$, you need the KEY identity and its first derivative

$$\begin{aligned} \mathrm{L}(e^{zt}) &= (z^2 - 2z + 5)e^{zt} , \\ \mathrm{L}(t\,e^{zt}) &= (z^2 - 2z + 5)t\,e^{zt} + (2z - 2)\,e^{zt} . \end{aligned}$$

Evaluate these at $z = 1$ to find $\mathrm{L}(e^t) = 4e^t$ and $\mathrm{L}(t\,e^t) = 4t\,e^t$. Dividing the second of these equations by 4 yields $\mathrm{L}(\frac{1}{4}t\,e^t) = t\,e^t$, which implies $y_{P1}(t) = \frac{1}{4}t\,e^t$. The forcing term $\cos(2t)$ has degree $d = 0$ and characteristic $r + is = i2$, which is a root of $p(z)$ of multiplicity $m = 0$. Because $m + d = 0$, you only need the KEY identity

$$\mathrm{L}(e^{zt}) = (z^2 - 2z + 5)e^{zt} .$$

Evaluate this at $z = i2$ to find $\mathrm{L}(e^{i2t}) = (1 - i4)e^{i2t}$. Dividing this by $(1 - i4)$ yeilds

$$\mathrm{L}\left(\frac{e^{i2t}}{1 - i4}\right) = e^{i2t} .$$

Because $\cos(2t) = \mathrm{Re}(e^{i2t})$, the above equation implies

$$\begin{aligned} y_{P2}(t) &= \mathrm{Re}\left(\frac{e^{i2t}}{1 - i4}\right) = \mathrm{Re}\left(\frac{(1 + i4)e^{i2t}}{1^2 + 4^2}\right) \\ &= \tfrac{1}{17}\,\mathrm{Re}\big((1 + i4)e^{i2t}\big) = \tfrac{1}{17}\big(\cos(2t) - 4\sin(2t)\big) . \end{aligned}$$

Combining these particular solutions with the general solution of the associated homogeneous problem yields the general solution

$$\begin{aligned} y &= y_H(t) + y_{P1}(t) + y_{P2}(t) \\ &= c_1 e^t \cos(2t) + c_2 e^t \sin(2t) + \tfrac{1}{4}t\,e^t + \tfrac{1}{17}\cos(2t) - \tfrac{4}{17}\sin(2t) . \end{aligned}$$

Direct Substitution. The forcing term $t\,e^t$ has degree $d = 1$ and characteristic $r + is = 1$, which is a root of $p(z)$ of multiplicity $m = 0$. Because $m = 0$ and $m + d = 1$, you seek a particular solution of the form

$$y_{P1}(t) = A_0 t\, e^t + A_1 e^t \,.$$

Because

$$y'_{P1}(t) = A_0 t\, e^t + (A_0 + A_1)e^t \,, \qquad y''_{P1}(t) = A_0 t\, e^t + (2A_0 + A_1)e^t \,,$$

one sees that

$$\begin{aligned} \mathrm{L}y_{P1}(t) &= y''_{P1}(t) - 2y'_{P1}(t) + 5y_{P1}(t) \\ &= \big(A_0 t\, e^t + (2A_0 + A_1)e^t\big) - 2\big(A_0 t\, e^t + (A_0 + A_1)e^t\big) \\ &\quad + 5\big(A_0 t\, e^t + A_1 e^t\big) \\ &= 4A_0 t\, e^t + 4A_1 e^t \,. \end{aligned}$$

Setting $4A_0 t\, e^t + 4A_1 e^t = t\, e^t$, we see that $4A_0 = 1$ and $4A_1 = 0$, whereby $A_0 = \frac{1}{4}$ and $A_1 = 0$. Hence, $y_P(t) = \frac{1}{4} t\, e^t$.
The forcing term $\cos(2t)$ has degree $d = 0$ and characteristic $r + is = i2$, which is a root of $p(z)$ of multiplicity $m = 0$. Because $m = 0$ and $m + d = 0$, you seek a particular solution of the form

$$y_{P2}(t) = A\cos(2t) + B\sin(2t) \,.$$

Because

$$\begin{aligned} y'_{P2}(t) &= -2A\sin(2t) + 2B\cos(2t) \,, \\ y''_{P2}(t) &= -4A\cos(2t) - 4B\sin(2t) \,, \end{aligned}$$

one sees that

$$\begin{aligned} \mathrm{L}y_{P2}(t) &= y''_{P2}(t) - 2y'_{P2}(t) + 5y_{P2}(t) \\ &= \big(-4A\cos(2t) - 4B\sin(2t)\big) - 2\big(-2A\sin(2t) + 2B\cos(2t)\big) \\ &\quad + 5\big(A\cos(2t) + B\sin(2t)\big) \\ &= (A - 4B)\cos(2t) + (B + 4A)\sin(2t) \,. \end{aligned}$$

Setting $(A - 4B)\cos(2t) + (B + 4A)\sin(2t) = \cos(2t)$, we see that

$$A - 4B = 1 \,, \qquad B + 4A = 0 \,.$$

This system can be solved by any method you choose to find $A = \frac{1}{17}$ and $B = -\frac{4}{17}$, whereby

$$y_{P2}(t) = \tfrac{1}{17}\cos(2t) - \tfrac{4}{17}\sin(2t) \,.$$

Combining these particular solutions with the general solution of the associated homogeneous problem yields the general solution

$$\begin{aligned} y &= y_H(t) + y_{P1}(t) + y_{P2}(t) \\ &= c_1 e^t\cos(2t) + c_2 e^t \sin(2t) + \tfrac{1}{4} t\, e^t + \tfrac{1}{17}\cos(2t) - \tfrac{4}{17}\sin(2t) \,. \end{aligned}$$

(b) $\dfrac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 9y = \tan(3t)$

Solution. This is a constant coefficient, inhomogeneous, linear equation. Its characteristic polynomial is

$$p(z) = z^2 + 9 = z^2 + 3^2 .$$

This has the conjugate pair of roots $\pm i3$, which yields a general solution of the associated homogeneous problem

$$y_H(t) = c_1 \cos(3t) + c_2 \sin(3t) .$$

Because of the form of the forcing term, you must use either the Green function method or variation of parameters to find a particular solution.

Variation of Parameters. The equation is already in normal form. Seek a solution in the form

$$y(t) = u_1(t)\cos(3t) + u_2(t)\sin(3t) ,$$

where $u_1'(t)$ and $u_2'(t)$ satisfy

$$\begin{aligned} u_1'(t)\cos(3t) + u_2'(t)\sin(3t) &= 0 , \\ -u_1'(t)3\sin(3t) + u_2'(t)3\cos(3t) &= \tan(3t) . \end{aligned}$$

Solve this system to find

$$u_1'(t) = -\frac{\sin(3t)^2}{3\cos(3t)} = \tfrac{1}{3}\cos(3t) - \tfrac{1}{3}\sec(3t) , \qquad u_2'(t) = \tfrac{1}{3}\sin(3t) .$$

Integrate these to find

$$u_1(t) = c_1 + \tfrac{1}{9}\sin(3t) - \tfrac{1}{9}\log\bigl(\tan(3t) + \sec(3t)\bigr) , \qquad u_2(t) = c_2 - \tfrac{1}{9}\cos(3t) .$$

A general solution is therefore

$$\begin{aligned} y &= u_1(t)\cos(3t) + u_2(t)\sin(3t) \\ &= c_1\cos(3t) + c_2\sin(3t) - \tfrac{1}{9}\cos(3t)\log\bigl(\tan(3t) + \sec(3t)\bigr) . \end{aligned}$$

Green Function Method. The associated Green function $g(t)$ satisfies the initial-value problem

$$\frac{\mathrm{d}^2 g}{\mathrm{d}t^2} + 9g = 0 , \qquad g(0) = 0 , \quad g'(0) = 1 .$$

Because $g(t) = c_1\cos(3t) + c_2\sin(3t)$, the first initial condition implies $c_1 = g(0) = 0$. Because then $g'(t) = 3c_2\cos(3t)$, the second initial condition implies $3c_2 = g'(0) = 1$. Hence,

$$g(t) = \tfrac{1}{3}\sin(3t) .$$

A particular solution is then given by

$$\begin{aligned} y_P(t) &= \int_0^t g(t-s)\tan(3s)\,\mathrm{d}s = \tfrac{1}{3}\int_0^t \sin(3t-3s)\tan(3s)\,\mathrm{d}s \\ &= \tfrac{1}{3}\int_0^t \big(\sin(3t)\cos(3s)-\cos(3t)\sin(3s)\big)\tan(3s)\,\mathrm{d}s \\ &= \tfrac{1}{3}\sin(3t)\int_0^t \sin(3s)\,\mathrm{d}s - \tfrac{1}{3}\cos(3t)\int_0^t \frac{\sin(3s)^2}{\cos(3s)}\,\mathrm{d}s\,. \end{aligned}$$

Because

$$\begin{aligned} \int_0^t \sin(3s)\,\mathrm{d}s &= -\tfrac{1}{3}\cos(3s)\Big|_{s=0}^{t} = \tfrac{1}{3}-\tfrac{1}{3}\cos(3t)\,, \\ \int_0^t \frac{\sin(3s)^2}{\cos(3s)}\,\mathrm{d}s &= \int_0^t \sec(3s)-\cos(3s)\,\mathrm{d}s \\ &= \tfrac{1}{3}\Big(\log\big(\tan(3s)+\sec(3s)\big)-\sin(3s)\Big)\Big|_{s=0}^{t} \\ &= \tfrac{1}{3}\Big(\log\big(\tan(3t)+\sec(3t)\big)-\sin(3t)\Big)\,, \end{aligned}$$

you find that

$$\begin{aligned} y_P(t) &= \tfrac{1}{9}\sin(3t)\big(1-\cos(3t)\big) - \tfrac{1}{9}\cos(3t)\Big(\log\big(\tan(3t)+\sec(3t)\big)-\sin(3t)\Big) \\ &= \tfrac{1}{9}\sin(3t) - \tfrac{1}{9}\cos(3t)\log\big(\tan(3t)+\sec(3t)\big)\,. \end{aligned}$$

A general solution is therefore

$$\begin{aligned} y &= y_H(t) + y_P(t) \\ &= c_1\cos(3t) + c_2\sin(3t) + \tfrac{1}{9}\sin(3t) - \tfrac{1}{9}\cos(3t)\log\big(\tan(3t)+\sec(3t)\big)\,. \end{aligned}$$

(5) When a mass of 2 kilograms is hung vertically from a spring, it stretches the spring 0.5 meters. (Gravitational acceleration is 9.8 m/sec^2.) At $t = 0$ the mass is set in motion from 0.3 meters below its equilibrium (rest) position with a upward velocity of 2 m/sec. Neglect drag and assume that the spring force is proportional to its displacement. Formulate an initial-value problem that governs the motion of the mass for $t > 0$. (DO NOT solve this initial-value problem; just write it down!)

Solution. Let $h(t)$ be the displacement (in meters) of the mass from its equilibrium (rest) position at time t (in seconds), with upward displacements being positive. The governing initial-value problem then has the form

$$m\frac{\mathrm{d}^2h}{\mathrm{d}t^2} + kh = 0\,, \qquad h(0) = -.3\,, \quad h'(0) = 2\,,$$

where m is the mass and k is the spring constant. The problem says that $m = 2$ kilograms. The spring constant is obtained by balancing the weight of the mass (mg $= 2 \cdot 9.8$ Newtons) with the force applied by the spring when it is stetched .5 m. This gives $k\,.5 = 2 \cdot 9.8$, or

$$k = \frac{2\cdot 9.8}{.5} = 4\cdot 9.8 \quad \text{Newtons/m}\,.$$

The governing initial-value problem is therefore

$$2\frac{\mathrm{d}^2h}{\mathrm{d}t^2} + 4 \cdot 9.8h = 0\,, \qquad h(0) = -.3\,, \quad h'(0) = 2\,.$$

Had you chosen positive h to be downward displacements then the only thing that would differ is the sign of the initial data.

(6) Give an explicit general solution of the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2\frac{\mathrm{d}y}{\mathrm{d}t} + 5y = 0\,.$$

Sketch a typical solution for $t \geq 0$. If this equation governs a damped spring-mass system, is the system over, under, or critically damped?

Solution. This is a constant coefficient, homogeneous, linear equation. Its characteristic polynomial is

$$p(z) = z^2 + 2z + 5 = (z+1)^2 + 2^2\,.$$

This has the conjugate pair of roots $-1 \pm i2$, which yields a general solution

$$y = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)\,.$$

When $c_1^2 + c_2^2 > 0$ this can be put into the amplitute-phase form

$$y = Ae^{-t}\cos(2t - \delta)\,,$$

where $A > 0$ and $0 \leq \delta < 2\pi$ are determined from c_1 and c_2 by

$$A = \sqrt{c_1^2 + c_2^2}\,, \qquad \cos(\delta) = \frac{c_1}{A}\,, \qquad \sin(\delta) = \frac{c_2}{A}\,.$$

In other words, (A, δ) are the polar coordinates for the point in the plane whose Cartesian coordinates are (c_1, c_2). The sketch should show a decaying oscillation with amplitude Ae^{-t} and quasiperiod $\frac{2\pi}{2} = \pi$. The equation governs an *under damped* spring-mass system because its characteristic polynomial has a conjugate pair of roots.

(7) Find the Laplace transform $Y(s)$ of the solution $y(t)$ to the initial-value problem

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 4\frac{\mathrm{d}y}{\mathrm{d}t} + 8y = f(t)\,, \qquad y(0) = 2\,, \quad y'(0) = 4\,.$$

where

$$f(t) = \begin{cases} 4 & \text{for } 0 \leq t < 2\,, \\ t^2 & \text{for } 2 \leq t\,. \end{cases}$$

You may refer to the table in Section 6.2 of the book. (DO NOT take the inverse Laplace transform to find $y(t)$; just solve for $Y(s)$!)

Solution. The Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) + 4\mathcal{L}[y'](s) + 8\mathcal{L}[y](s) = \mathcal{L}[f](s)\,,$$

where

$$\begin{aligned} \mathcal{L}[y](s) &= Y(s)\,, \\ \mathcal{L}[y'](s) &= sY(s) - y(0) = sY(s) - 2\,, \\ \mathcal{L}[y''](s) &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2s - 4\,. \end{aligned}$$

To compute $\mathcal{L}[f](s)$, first write f as

$$\begin{aligned} f(t) &= \big(1-u(t-2)\big)4 + u(t-2)t^2 = 4 - u(t-2)4 + u(t-2)t^2 \\ &= 4 + u(t-2)(t^2-4) = 4 + u(t-2)\Big(\big(2+(t-2)\big)^2 - 4\Big) \\ &= 4 + u(t-2)\big(4(t-2) + (t-2)^2\big) . \end{aligned}$$

Referring to the table of Laplace transforms in the book, item 13 with $c = 2$, item 1, and item 3 with $n = 1$ and $n = 2$ then show that

$$\begin{aligned} \mathcal{L}[f](s) &= 4\mathcal{L}[1](s) + 4\mathcal{L}\big[u(t-2)(t-2)\big](s) + \mathcal{L}\big[u(t-2)(t-2)^2\big](s) \\ &= 4\mathcal{L}[1](s) + 4e^{-2s}\mathcal{L}[t](s) + e^{-2s}\mathcal{L}[t^2](s) \\ &= 4\frac{1}{s} + 4e^{-2s}\frac{1}{s^2} + e^{-2s}\frac{2}{s^3} . \end{aligned}$$

The Laplace transform of the initial-value problem then becomes

$$\big(s^2Y(s) - 2s - 4\big) + 4\big(sY(s) - 2\big) + 8Y(s) = \frac{4}{s} + e^{-2s}\frac{4}{s^2} + e^{-2s}\frac{2}{s^3} ,$$

which becomes

$$(s^2+4s+8)Y(s) - 2s - 12 = \frac{4}{s} + e^{-2s}\frac{4}{s^2} + e^{-2s}\frac{2}{s^3} .$$

Hence, $Y(s)$ is given by

$$Y(s) = \frac{1}{s^2+4s+8}\left(2s + 12 + \frac{4}{s} + e^{-2s}\frac{4}{s^2} + e^{-2s}\frac{2}{s^3}\right) .$$

(8) Find the function $y(t)$ whose Laplace transform $Y(s)$ is given by

$$\text{(a)} \quad Y(s) = \frac{e^{-3s}4}{s^2-6s+5} , \qquad\qquad \text{(b)} \quad Y(s) = \frac{e^{-2s}s}{s^2+4s+8} .$$

You may refer to the table in Section 6.2 of the book.

Solution (a). The denominator factors as $(s-5)(s-1)$, so the partial fraction decomposition is

$$\frac{4}{s^2-6s+5} = \frac{4}{(s-5)(s-1)} = \frac{1}{s-5} - \frac{1}{s-1} .$$

Referring to the table of Laplace transforms in the book, item 11 with $n = 0$ and $a = 5$, and with $n = 0$ and $a = 1$ gives

$$\mathcal{L}[e^{5t}](s) = \frac{1}{s-5} , \qquad \mathcal{L}[e^{t}](s) = \frac{1}{s-1} ,$$

whereby

$$\frac{4}{s^2-6s+5} = \mathcal{L}[e^{5t}](s) - \mathcal{L}[e^{t}](s) = \mathcal{L}\big[e^{5t} - e^{t}\big](s) .$$

It follows from item 13 with $c = 3$ and $f(t) = e^{5t} - e^t$ that

$$\mathcal{L}\big[u(t-3)\big(e^{5(t-3)} - e^{t-3}\big)\big](s) = e^{-3s}\frac{4}{s^2-6s+5} = Y(s) .$$

You therefore conclude that

$$y(t) = \mathcal{L}^{-1}[Y(s)](t) = u(t-3)\big(e^{5(t-3)} - e^{t-3}\big)\,.$$

Solution (b). The denominator does not have real factors. The partial fraction decomposition is

$$\frac{s}{s^2+4s+8} = \frac{s}{(s+2)^2+4} = \frac{s+2}{(s+2)^2+2^2} - \frac{2}{(s+2)^2+2^2}\,.$$

Referring to the table of Laplace transforms in the book, items 10 and 9 with $a = -2$ and $b = 2$ give

$$\mathcal{L}[e^{-2t}\cos(2t)](s) = \frac{s+2}{(s+2)^2+2^2}\,, \qquad \mathcal{L}[e^{-2t}\sin(2t)](s) = \frac{2}{(s+2)^2+2^2}\,,$$

whereby

$$\begin{aligned}\frac{s}{s^2+4s+8} &= \mathcal{L}[e^{-2t}\cos(2t)](s) - \mathcal{L}[e^{-2t}\sin(2t)](s)\\ &= \mathcal{L}\big[e^{-2t}\big(\cos(2t)-\sin(2t)\big)\big](s)\,.\end{aligned}$$

It follows from item 13 with $c = 2$ and $f(t) = e^{-2t}\big(\cos(2t)-\sin(2t)\big)$ that

$$\mathcal{L}\big[u(t-2)e^{-2(t-2)}\big(\cos(2(t-2)) - \sin(2(t-2))\big)\big](s) = e^{-2s}\frac{s}{s^2+4s+8} = Y(s)\,.$$

You therefore conclude that

$$y(t) = \mathcal{L}^{-1}[Y(s)](t) = u(t-2)e^{-2(t-2)}\big(\cos(2(t-2)) - \sin(2(t-2))\big)\,.$$

(9) Consider the real vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix}1\\ t\end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix}t^3\\ 3+t^4\end{pmatrix}$.

(a) Compute the Wronskian $W[\mathbf{x}_1,\mathbf{x}_2](t)$.

Solution. The Wronskian is given by

$$W[\mathbf{x}_1,\mathbf{x}_2](t) = \det\begin{pmatrix}1 & t^3\\ t & 3+t^4\end{pmatrix} = 1\cdot(3+t^4) - t\cdot t^3 = 3 + t^4 - t^4 = 3\,.$$

(b) Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to the linear system $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x}$.

Solution. Set

$$\mathbf{\Psi}(t) = \big(\mathbf{x}_1(t)\ \ \mathbf{x}_2(t)\big) = \begin{pmatrix}1 & t^3\\ t & 3+t^4\end{pmatrix}\,.$$

If $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions to the linear system then the matrix-valued function $\mathbf{\Psi}$ must satisfy

$$\frac{\mathrm{d}\mathbf{\Psi}}{\mathrm{d}t}(t) = \mathbf{A}(t)\mathbf{\Psi}(t)\,.$$

Because $\det(\mathbf{\Psi}(t)) = W[\mathbf{x}_1, \mathbf{x}_2](t) = 3 \neq 0$, we see that $\mathbf{\Psi}(t)$ is a fundamental matrix of the linear system with $\mathbf{A}(t)$ given by

$$\begin{aligned}\mathbf{A}(t) &= \frac{\mathrm{d}\mathbf{\Psi}}{\mathrm{d}t}(t)\,\mathbf{\Psi}(t)^{-1} = \begin{pmatrix} 0 & 3t^2 \\ 1 & 4t^3 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 3+t^4 & -t^3 \\ -t & 1 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} -3t^3 & 3t^2 \\ 3-3t^3 & 3t^3 \end{pmatrix} = \begin{pmatrix} -t^3 & t^2 \\ 1-t^3 & t^3 \end{pmatrix}.\end{aligned}$$

It follows that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to the linear system with this $\mathbf{A}(t)$.

(c) Give a general solution to the system you found in part (b).

Solution. Because $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to the linear system with the above $\mathbf{A}(t)$, a general solution is given by

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1 \begin{pmatrix} 1 \\ t \end{pmatrix} + c_2 \begin{pmatrix} t^3 \\ 3+t^4 \end{pmatrix}.$$

(10) Give a general real vector-valued solution of the linear planar system $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x}$ for

(a) $\mathbf{A} = \begin{pmatrix} 6 & 4 \\ 4 & 0 \end{pmatrix}$, (b) $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$.

Solution (a). The characteristic polynomial of $\mathbf{A}$ is

$$\begin{aligned}p(z) &= z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) \\ &= z^2 - 6z - 16 = (z-3)^2 - 25 = (z-3)^2 - 5^2.\end{aligned}$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are 3 ± 5, or simply -2 and 8. One therefore has

$$\begin{aligned}e^{t\mathbf{A}} &= e^{3t}\left[\mathbf{I}\cosh(5t) + (\mathbf{A} - 3\mathbf{I})\frac{\sinh(5t)}{5}\right] \\ &= e^{3t}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cosh(5t) + \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\frac{\sinh(5t)}{5}\right] \\ &= e^{3t}\begin{pmatrix} \cosh(5t) + \frac{3}{5}\sinh(5t) & \frac{4}{5}\sinh(5t) \\ \frac{4}{5}\sinh(5t) & \cosh(5t) - \frac{3}{5}\sinh(5t) \end{pmatrix}.\end{aligned}$$

A general solution is therefore given by

$$\mathbf{x} = c_1 e^{3t}\begin{pmatrix} \cosh(5t) + \frac{3}{5}\sinh(5t) \\ \frac{4}{5}\sinh(5t) \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} \frac{4}{5}\sinh(5t) \\ \cosh(5t) - \frac{3}{5}\sinh(5t) \end{pmatrix}.$$

Alternative Solution (a). The characteristic polynomial of $\mathbf{A}$ is

$$\begin{aligned}p(z) &= z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) \\ &= z^2 - 6z - 16 = (z-3)^2 - 25 = (z-3)^2 - 5^2.\end{aligned}$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are 3 ± 5, or simply -2 and 8. Because

$$\mathbf{A} + 2\mathbf{I} = \begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix}, \qquad \mathbf{A} - 8\mathbf{I} = \begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix},$$

we see that $\mathbf{A}$ has the eigenpairs

$$\left(-2, \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right), \qquad \left(8, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right).$$

A general solution is therefore given by

$$\mathbf{x} = c_1 e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{8t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Solution (b). The characteristic polynomial of $\mathbf{A}$ is

$$\begin{aligned} p(z) &= z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) \\ &= z^2 - 2z + 5 = (z-1)^2 + 4 = (z-1)^2 + 2^2. \end{aligned}$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $1 \pm i2$. One therefore has

$$\begin{aligned} e^{t\mathbf{A}} &= e^t \left[\mathbf{I}\cos(2t) + (\mathbf{A} - \mathbf{I})\frac{\sin(2t)}{2} \right] \\ &= e^t \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(2t) + \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \frac{\sinh(2t)}{2} \right] \\ &= e^t \begin{pmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{pmatrix}. \end{aligned}$$

A general solution is therefore given by

$$\mathbf{x} = c_1 e^t \begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix}.$$

Alternative Solution (b). The characteristic polynomial of $\mathbf{A}$ is

$$\begin{aligned} p(z) &= z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) \\ &= z^2 - 2z + 5 = (z-1)^2 + 4 = (z-1)^2 + 2^2. \end{aligned}$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $1 \pm i2$. Because

$$\mathbf{A} - (1+i2)\mathbf{I} = \begin{pmatrix} -i2 & 2 \\ -2 & -i2 \end{pmatrix}, \qquad \mathbf{A} - (1-i2)\mathbf{I} = \begin{pmatrix} i2 & 2 \\ -2 & i2 \end{pmatrix},$$

we see that $\mathbf{A}$ has the eigenpairs

$$\left(1+i2, \begin{pmatrix} 1 \\ i \end{pmatrix}\right), \qquad \left(1-i2, \begin{pmatrix} -i \\ 1 \end{pmatrix}\right).$$

Because

$$e^{(1+i2)t} \begin{pmatrix} 1 \\ i \end{pmatrix} = e^t \begin{pmatrix} \cos(2t) + i\sin(2t) \\ -\sin(2t) + i\cos(2t) \end{pmatrix},$$

two real solutions of the system are

$$e^t \begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix}, \qquad e^t \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix}.$$

A general solution is therefore

$$\mathbf{x} = c_1 e^t \begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix}.$$

(11) A real 2×2 matrix $\mathbf{A}$ has eigenvalues 2 and -1 with associated eigenvectors

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

(a) Give a general solution to the linear planar system $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x}$.

Solution. A general solution is

$$\mathbf{x} = c_1 e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

(b) Classify the stability of the origin. Sketch a phase-plane portrait for this system and identify its type. (Carefully mark all sketched trajectories with arrows!)

Solution. The coefficient matrix has two real eigenvalues of opposite sign. The origin is therefore a *saddle* and is thereby *unstable*. There is one trajectory moves away from $(0,0)$ along each half of the line $x = 3y$, and one trajectory moves towards$(0,0)$ along each half of the line $y = -2x$. (These are the lines of eigenvectors.) Every other trajectory sweeps away from the line $y = -2x$ and towards the line $x = 3y$. A phase-plane portrait was sketched during the review session.

(12) Consider the nonlinear planar system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -5y\,,$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = x - 4y - x^2\,.$$

(a) Find all of its equilibrium (critical, stationary) points.

Solution. Stationary points satisfy

$$0 = -5y\,, \qquad 0 = x - 4y - x^2\,.$$

The first equation implies $y = 0$, whereby the second equation becomes $0 = x - x^2 = x(1-x)$, which implies either $x = 0$ or $x = 1$. All the stationary points of the system are therefore

$$(0,0)\,, \qquad (1,0)\,.$$

(b) Compute the coefficient matrix of the linearization (the derivative matrix) at each equilibrium (critical, stationary) point.

Solution. Because

$$\begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = \begin{pmatrix} -5y \\ x - 4y - x^2 \end{pmatrix},$$

the matrix of partial derivatives is

$$\begin{pmatrix} \partial_x f(x,y) & \partial_y f(x,y) \\ \partial_x g(x,y) & \partial_y g(x,y) \end{pmatrix} = \begin{pmatrix} 0 & -5 \\ 1-2x & -4 \end{pmatrix}.$$

Evaluating this matrix at each stationary point yields the coefficient matrices

$$\mathbf{A} = \begin{pmatrix} 0 & -5 \\ 1 & -4 \end{pmatrix} \quad \text{at } (0,0), \qquad \mathbf{A} = \begin{pmatrix} 0 & -5 \\ -1 & -4 \end{pmatrix} \quad \text{at } (1,0).$$

(c) Classify the type and stability of each equilibrium (critical, stationary) point.

Solution. The coefficient matrix $\mathbf{A}$ at $(0,0)$ has eigenvalues that satisfy

$$0 = \det(z\mathbf{I} - \mathbf{A}) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 4z + 5 = (z+2)^2 + 1^2.$$

The eigenvalues are thereby $-2 \pm i$. Because $a_{21} = 1 > 0$, the stationary point $(0,0)$ is therefore a *counterclockwise spiral sink*, which is *asymptotically stable* or *attracting*. This is one of the generic types, so it describes the phase-plane portrait of the nonlinear system near $(0,0)$.

The coefficient matrix $\mathbf{A}$ at $(1,0)$ has eigenvalues that satisfy

$$0 = \det(z\mathbf{I} - \mathbf{A}) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 4z - 5 = (z+2)^2 - 3^2.$$

The eigenvalues are thereby -2 ± 3, or simply -5 and 1. The stationary point $(1,0)$ is therefore a *saddle*, which is *unstable*. This is one of the generic types, so it describes the phase-plane portrait of the nonlinear system near $(1,0)$.

(d) Sketch a plausible global phase-plane portrait. (Carefully mark all sketched trajectories with arrows!)

Solution. The nullcline for $\frac{\mathrm{d}x}{\mathrm{d}t}$ is the line $y = 0$. This line partitions the plane into regions where x is increasing or decreasing as t increases. The nullcline for $\frac{\mathrm{d}y}{\mathrm{d}t}$ is the parabola $y = \frac{1}{4}(x - x^2)$. This curve partitions the plane into regions where y is increasing or decreasing as t increases. Neither of these nullclines is invariant.

The stationary point $(0,0)$ is a *counterclockwise spiral sink.*

The stationary point $(1,0)$ is a *saddle.* The coefficient matrix $\mathbf{A}$ has eigenvalues -5 and 1. Because

$$\mathbf{A} + 5\mathbf{I} = \begin{pmatrix} 5 & -5 \\ -1 & 1 \end{pmatrix}, \qquad \mathbf{A} - \mathbf{I} = \begin{pmatrix} -1 & -5 \\ -1 & -5 \end{pmatrix},$$

it has the eigenpairs

$$\left(-5, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right), \qquad \left(1, \begin{pmatrix} -5 \\ 1 \end{pmatrix}\right)$$

Near $(1,0)$ there is one trajectory that emerges from $(1,0)$ tangent to each side of the line $x = 1-5y$. There is also one trajectory that approaches $(1,0)$ tangent to each side of the line $y = x-1$. These trajectories are separatrices. A global phase-plane portrait was sketched during the review session.

Remark. The global phase-plane portrait becomes clearer if you are able to observe that $H(x,y) = \frac{1}{2}x^2 + \frac{5}{2}y^2 - \frac{1}{3}x^3$ satisfies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}H(x,y) &= \partial_x H(x,y)\frac{\mathrm{d}x}{\mathrm{d}t} + \partial_y H(x,y)\frac{\mathrm{d}y}{\mathrm{d}t} \\ &= (x-x^2)(-5y) + 5y(x-4y-x^2) = -20y^2 \le 0\,.\end{aligned}$$

The trajectories of the system are thereby seen to cross the level sets of $H(x,y)$ so as to decrease $H(x,y)$. You would not be expected to see this on the Final.

(13) Consider the nonlinear planar system

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= x(3-3x+2y)\,, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= y(6-x-y)\,.\end{aligned}$$

Do parts (a-d) as for the previous problem.

(a) Find all of its equilibrium (critical, stationary) points.

Solution. Stationary points satisfy

$$0 = x(3-3x+2y)\,, \qquad 0 = y(6-x-y)\,.$$

The first equation implies either $x=0$ or $3-3x+2y=0$, while the second equation implies either $y=0$ or $6-x-y=0$. If $x=0$ and $y=0$ then $(0,0)$ is a stationary point. If $x=0$ and $6-x-y=0$ then $(0,6)$ is a stationary point. If $3-3x+2y=0$ and $y=0$ then $(1,0)$ is a stationary point. If $3-3x+2y=0$ and $6-x-y=0$ then upon solving these equations one finds that $(3,3)$ is a stationary point. All the stationary points of the system are therefore

$$(0,0)\,, \qquad (0,6)\,, \qquad (1,0)\,, \qquad (3,3)\,.$$

(b) Compute the coefficient matrix of the linearization (the derivative matrix) at each equilibrium (critical, stationary) point.

Solution. Because

$$\begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = \begin{pmatrix} 3x-3x^2+2xy \\ 6y-xy-y^2 \end{pmatrix}\,,$$

the matrix of partial derivatives is

$$\begin{pmatrix} \partial_x f(x,y) & \partial_y f(x,y) \\ \partial_x g(x,y) & \partial_y g(x,y) \end{pmatrix} = \begin{pmatrix} 3-6x+2y & 2x \\ -y & 6-x-2y \end{pmatrix}\,.$$

Evaluating this matrix at each stationary point yields the coefficient matrices

$$\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix} \quad \text{at } (0,0)\,, \qquad \mathbf{A} = \begin{pmatrix} 15 & 0 \\ -6 & -6 \end{pmatrix} \quad \text{at } (0,6)\,,$$

$$\mathbf{A} = \begin{pmatrix} -3 & 2 \\ 0 & 5 \end{pmatrix} \quad \text{at } (1,0)\,, \qquad \mathbf{A} = \begin{pmatrix} -9 & 6 \\ -3 & -3 \end{pmatrix} \quad \text{at } (3,3)\,.$$

(c) Identify the type and stability of each equilibrium (critical, stationary) point.

Solution. The coefficient matrix $\mathbf{A}$ at $(0,0)$ is diagonal, so you can read-off its eigenvalues as 3 and 6. The stationary point $(0,0)$ is thereby a *nodal source*, which is *unstable* (or even better is *repelling*). This is one of the generic types, so it describes the phase-plane portrait of the nonlinear system near $(0,0)$.

The coefficient matrix $\mathbf{A}$ at $(0,6)$ is triangular, so you can read-off its eigenvalues as -6 and 15. The stationary point $(0,6)$ is thereby a *saddle*, which is *unstable*. This is one of the generic types, so it describes the phase-plane portrait of the nonlinear system near $(0,6)$.

The coefficient matrix $\mathbf{A}$ at $(1,0)$ is triangular, so you can read-off its eigenvalues as -3 and 5. The stationary point $(1,0)$ is thereby a *saddle*, which is *unstable*. This is one of the generic types, so it describes the phase-plane portrait of the nonlinear system near $(1,0)$.

The coefficient matrix $\mathbf{A}$ at $(0,6)$ has eigenvalues that satisfy

$$0 = \det(z\mathbf{I} - \mathbf{A}) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 12z + 45 = (z+6)^2 + 3^2\,.$$

Its eigenvalues are thereby $-6 \pm i3$. Because $a_{21} = -3 < 0$, the stationary point $(3,3)$ is therefore a *clockwise spiral sink*, which is *asymptotically stable* or *attracting*. This is one of the generic types, so it describes the phase-plane portrait of the nonlinear system near $(3,3)$.

(d) Sketch a plausible global phase-plane portrait. (Carefully mark all sketched trajectories with arrows!)

Solution. The nullclines for $\frac{dx}{dt}$ are the lines $x = 0$ and $3 - 3x + 2y = 0$. These lines partition the plane into regions where x is increasing or decreasing as t increases. The nullclines for $\frac{dy}{dt}$ are the lines $y = 0$ and $6 - x - y = 0$. These lines partition the plane into regions where y is increasing or decreasing as t increases.

Next, observe that the lines $x = 0$ and $y = 0$ are invariant. A trajectory that starts on one of these lines must stay on that line. Along the line $x = 0$ the system reduces to

$$\frac{dy}{dt} = y(6-y)\,.$$

Along the line $y = 0$ the system reduces to

$$\frac{dx}{dt} = 3x(1-x)\,.$$

The arrows along these invariant lines can be determined from a phase-line portrait of these reduced systems.

The stationary point $(0,0)$ is a *nodal source*. The coefficient matrix $\mathbf{A}$ has eigenvalues 3 and 6. Because

$$\mathbf{A} - 3\mathbf{I} = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}, \qquad \mathbf{A} - 6\mathbf{I} = \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix},$$

it has the eigenpairs

$$\left(3\,, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right), \qquad \left(6\,, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

Near there is one trajectory that emerges from $(0,0)$ along each side of the invariant lines $y = 0$ and $x = 0$. Every other trajectory emerges from $(0,0)$ tangent to the line $y = 0$, which is the line corresponding to the eigenvalue with the smaller absolute value.

The stationary point $(0,6)$ is a *saddle*. The coefficient matrix $\mathbf{A}$ has eigenvalues -6 and 15. Because

$$\mathbf{A} + 6\mathbf{I} = \begin{pmatrix} 21 & 0 \\ -6 & 0 \end{pmatrix}, \qquad \mathbf{A} - 15\mathbf{I} = \begin{pmatrix} 0 & 0 \\ -6 & -21 \end{pmatrix},$$

it has the eigenpairs

$$\left(-6\,, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right), \qquad \left(15\,, \begin{pmatrix} 7 \\ -2 \end{pmatrix}\right)$$

Near $(0,6)$ there is one trajectory that approaches $(0,6)$ along each side of the invariant line $x = 0$. There is also one trajectory that emerges from $(0,6)$ tangent to each side of the line $y = 6 - \frac{2}{7}x$. These trajectories are separatrices.

The stationary point $(1,0)$ is a *saddle*. The coefficient matrix $\mathbf{A}$ has eigenvalues -3 and 5. Because

$$\mathbf{A} + 3\mathbf{I} = \begin{pmatrix} 0 & 2 \\ 0 & 8 \end{pmatrix}, \qquad \mathbf{A} - 5\mathbf{I} = \begin{pmatrix} -8 & 2 \\ 0 & 0 \end{pmatrix},$$

it has the eigenpairs

$$\left(-3\,, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right), \qquad \left(5\,, \begin{pmatrix} 1 \\ 4 \end{pmatrix}\right)$$

Near $(1,0)$ there is one trajectory that emerges from $(1,0)$ along each side of the invariant line $y = 0$. There is also one trajectory that approaches $(1,0)$ tangent to each side of the line $y = 4(x-1)$. These trajectories are also separatrices.

Finally, the stationary point $(3,3)$ is a *clockwise spiral sink*. All trajectories in the positive quadrant will spiral into it. A phase-plane global portrait was sketched during the review session.