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% Math 246 Extra Credit

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% Professor Levermore


%This script will determine the critical points and eigenvalues
%as well as plot the vector field, contours, and trajectories for
%the equation H(x,y) = 1/2*y^2 + 1/4*x^4 + 2/3*b*x^3 + 1/2*c*x^2
%using different values for b and c.
%The following is the system of differential equations to solve:
%dx/dt = Hy(x,y) = y
%dy/dt = -Hx(x,y) = -(x^3 + 2*b*x^2 + c*x)

syms x y

%Find Critical Points and Eigenvalues

for b = -2:0.25:2
    for c = 1

        dxdt = y;
        dydt = -(x^3 +2*b*x^2 + c*x);

        [x_crit] = solve('-(x^3 + 2*b*x^2 + c*x)=0');
        disp(['CRITICAL POINTS AND EIGENVALUES FOR b = ',num2str(b)])
        [x_crit] = subs([x_crit]);
        %We know that y will always be 0 at the critical points because
        %dx/dt = 0 only when y = 0
        y_crit = zeros(size(x_crit));

        disp('Critical Points:');
        disp([x_crit,y_crit])

        %Compute Jacobian to find coefficient matrix
        A = real(jacobian([dxdt dydt], [x y]));
        disp('Coefficient Matrix:')
        disp(A)

        %Compute eigenvalues of coefficient matrix
        eigvals = eig(A);
        disp('Eigenvalues:')
        disp(eigvals)

        %Display coefficient matrix at critical points
        disp(['Coefficient Matrix at [', num2str(x_crit(1)),'
',num2str(y_crit(1)),']:']);
        disp(double(subs(A, {x,y},{x_crit(1),y_crit(1)})))
        disp(['Coefficient Matrix at [', num2str(x_crit(2)),'
',num2str(y_crit(2)),']:']);
        disp(double(subs(A, {x,y},{x_crit(2),y_crit(2)})))
        disp(['Coefficient Matrix at [', num2str(x_crit(3)),'
',num2str(y_crit(3)),']:']);
        disp(double(subs(A, {x,y},{x_crit(3),y_crit(3)})))

        %Display eigenvalues at critical points
        disp(['Eigenvalues at [', num2str(x_crit(1)),' ',num2str(y_crit(1)),']:']);
        disp(double(subs(eigvals, {x,y},{x_crit(1),y_crit(1)})))
        disp(['Eigenvalues at [', num2str(x_crit(2)),' ',num2str(y_crit(2)),']:']);
        disp(double(subs(eigvals, {x,y},{x_crit(2),y_crit(2)})))
        disp(['Eigenvalues at [', num2str(x_crit(3)),' ',num2str(y_crit(3)),']:']);
        disp(double(subs(eigvals, {x,y},{x_crit(3),y_crit(3)})))

    end
end

%The above outputs for critical points and eigenvalues have been
%transferred to an Excel spreadsheet that is attached to the end
%of this document in order to easily display their values and to show what
%type of critical point each one is as well as its stability.

%There are three critical points for the equation, and the x values change
%based on the value of b (the y value is 0 for each critical points).
%When b < -1, there is one critical point at (0,0) that is a stable center,
%there is one critical point on the x axis that is between 0 and 1 and is
%an unstable saddle point, and there is one critical point on the x axis that
%is greater than 1 and is a stable center.
%When b = -1, there is one critical point at (0,0) that is a stable center
%and there are two critical points at (1,0) with the same eigenvalues,
%meaning that they have a multiplicity of 2, and these critical points are
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%stable zeros.
%When -1 < b < 0 or 0 < b < 1, there is one critical point at (0,0) that is a
stable
%center, and there are two complex critical points that are unstable saddle points.
%When b = 0, there is one critical point at (0,0) that is a stable center,
%there is one complex critical point at (i,0) that is an unstable saddle point, and
%there is one complex critical point at (-i,0) that is an unstable saddle point.
%When b = 1, there is one critical point at (0,0) that is a stable center
%and there are two critical points at (-1,0) with the same eigenvalues,
%meaning that they have a multiplicity of 2, and these critical points are
%stable zeros.
%When b > 1, there is one critical point at (0,0) that is a stable center,
%there is one critical point on the x axis that is between -1 and 0 and is
%an unstable saddle point, and there is one critical point on the x axis that
%is less than -1 and is a stable center.

%The reason that two of the critical points are complex when -1 < b < 1
%is because the dy/dt equation can be factored into -x(x^2 + 2*b*x + 1).
%To find the nonzero critical points you would use the quadratic formula on
%the portion of the equation inside the parentheses. This gets you
%x_crit = b +/- sqrt(b^2 - 1). Therefore, when b^2 < 1 there are two
%complex critical points.

%It is interesting to see how the location, type, and stability of the
%critical points change as b changes, especially when comparing these
%changes to the visual output of the graphs. It is obvious that b = 1
%and b = -1 are some kind of transition points. At these points, when moving
%from 0 towards increasingly negative or increasingly positive x values, the
%two nonzero critical points transition from complex conjugates to
%real numbers and transition from two unstable saddle points to one unstable
%saddle point and one stable center. The graphs show this transition very clearly.


%PLOT VECTOR FIELD

[x, y] = meshgrid(-10:.2:10,-10:.2:10);

for b = -2:.25:2
    for c = 1

        %Define right hand side of dx/dt
        dxdt_rhs = y;
        %Define right hand side of dy/dt
        dydt_rhs = -(x.^3 + 2*b.*x.^2 + c.*x);

        %Normalize vectors to have the same length
        norm = sqrt(dxdt_rhs.^2 + dydt_rhs.^2);
        %Plot vector field
        figure
        quiver(x, y, dxdt_rhs./norm, dydt_rhs./norm, .5);
        hold on
        axis tight
        xlabel('x')
        ylabel('y')
        axis([-5 5 -2.5 2.5])
        title(['Vector Field for b = ', num2str(b) ' and c =', num2str(c)])

    end
end


%PLOT CONTOURS

[x, y] = meshgrid(-8:.2:8,-8:.2:8);

for b = -2:.25:2
    for c = 1

        figure
        contour(x, y, (1/2).*y.^2 + (1/4).*x.^4 + (2/3)*b.*x.^3 + (1/2).*c.*x.^2,
100)
        xlabel('x')
        ylabel('y')
        axis([-8 8 -8 8])
        title(['Contours for b = ', num2str(b) ' and c =', num2str(c)])

    end
end
```

```
%PLOT TRAJECTORIES AND VECTOR FIELD

[x, y] = meshgrid(-10:.25:10,-10:.25:10);

for b = -2:0.25:2
    for c = 1

        figure
        %Define right hand side of dx/dt
        dxdt_rhs = y;
        %Define right hand side of dy/dt
        dydt_rhs = -(x.^3 + 2*b.*x.^2 + c.*x);
        %Normalize vectors to have the same length
        norm = sqrt(dxdt_rhs.^2 + dydt_rhs.^2);
        quiver(x, y, dxdt_rhs./norm, dydt_rhs./norm, .5);
        axis([-8 5 -8 8])
        xlabel('x')
        ylabel('y')
        title(['Trajectories and Vector Field for b = ', num2str(b) ' and c =',
num2str(c)])

        %Define right hand side of dx/dt and dy/dt as vectors in order to
        %solve differential equations with ODE45
        sys = @(t,vec) [vec(2);-(vec(1).^3 + 2*b.*vec(1).^2 + c.*vec(1))];
        %Solve and plot system of differential equations for x(0) = m and y(0) = n
        %for several values of m and n
        for m = -2:2
            for n = -2:2
                hold on
                %Solve dx/dt and dy/dt on interval 0 to 10 and output to a matrix
                [t,mtrx] = ode45(sys,[0 10],[m n]);
                %Plot above solutions
                plot(mtrx(:,1),mtrx(:,2),'g')
                %Solve dx/dt and dy/dt on interval -10 to 0 and output to a matrix
                [t,mtrx] = ode45(sys,[0 -10],[m n]);
                %Plot above solutions
                plot(mtrx(:,1),mtrx(:,2),'g')
            end
        end

    end
end

%END CODE
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```
CRITICAL POINTS AND EIGENVALUES FOR b = -2
Critical Points:
         0         0
    3.7321         0
    0.2679         0

Coefficient Matrix:
[                                    0,                                    1]
[ -3/2*x^2+4*x-1+1/2*conj(-3*x^2+8*x),                                    0]

Eigenvalues:
  1/2*(-4-6*x^2+16*x+2*conj(-3*x^2+8*x))^(1/2)
 -1/2*(-4-6*x^2+16*x+2*conj(-3*x^2+8*x))^(1/2)

Coefficient Matrix at [0 0]:
     0     1
    -1     0

Coefficient Matrix at [3.7321 0]:
         0    1.0000
  -12.9282         0

Coefficient Matrix at [0.26795 0]:
         0    1.0000
    0.9282         0

Eigenvalues at [0 0]:
   0.0000 + 1.0000i
  -0.0000 - 1.0000i

Eigenvalues at [3.7321 0]:
   0.0000 + 3.5956i
  -0.0000 - 3.5956i
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Eigenvalues at [0.26795 0]:
    0.9634
   -0.9634

CRITICAL POINTS AND EIGENVALUES FOR b = -1.75
Critical Points:
         0         0
    3.1861         0
    0.3139         0

Coefficient Matrix:
[                                     0,                                     1]
[ -3/2*x^2+7/2*x-1+1/2*conj(-3*x^2+7*x),                                     0]

Eigenvalues:
  1/2*(-4-6*x^2+14*x+2*conj(-3*x^2+7*x))^(1/2)
 -1/2*(-4-6*x^2+14*x+2*conj(-3*x^2+7*x))^(1/2)

Coefficient Matrix at [0 0]:
     0     1
    -1     0

Coefficient Matrix at [3.1861 0]:
         0    1.0000
   -9.1515         0

Coefficient Matrix at [0.31386 0]:
         0    1.0000
    0.9015         0

Eigenvalues at [0 0]:
   0.0000 + 1.0000i
  -0.0000 - 1.0000i

Eigenvalues at [3.1861 0]:
   0.0000 + 3.0251i
  -0.0000 - 3.0251i

Eigenvalues at [0.31386 0]:
    0.9495
   -0.9495

CRITICAL POINTS AND EIGENVALUES FOR b = -1.5
Critical Points:
         0         0
    2.6180         0
    0.3820         0

Coefficient Matrix:
[                                   0,                                   1]
[ -3/2*x^2+3*x-1+1/2*conj(-3*x^2+6*x),                                   0]

Eigenvalues:
  1/2*(-4-6*x^2+12*x+2*conj(-3*x^2+6*x))^(1/2)
 -1/2*(-4-6*x^2+12*x+2*conj(-3*x^2+6*x))^(1/2)

Coefficient Matrix at [0 0]:
     0     1
    -1     0

Coefficient Matrix at [2.618 0]:
         0    1.0000
   -5.8541         0

Coefficient Matrix at [0.38197 0]:
         0    1.0000
    0.8541         0

Eigenvalues at [0 0]:
   0.0000 + 1.0000i
  -0.0000 - 1.0000i

Eigenvalues at [2.618 0]:
   0.0000 + 2.4195i
  -0.0000 - 2.4195i

Eigenvalues at [0.38197 0]:
    0.9242
   -0.9242
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```
CRITICAL POINTS AND EIGENVALUES FOR b = -1.25
Critical Points:
         0         0
    2.0000         0
    0.5000         0

Coefficient Matrix:
[                                      0,                                     1]
[ -3/2*x^2+5/2*x-1+1/2*conj(-3*x^2+5*x),                                     0]

Eigenvalues:
  1/2*(-4-6*x^2+10*x+2*conj(-3*x^2+5*x))^(1/2)
 -1/2*(-4-6*x^2+10*x+2*conj(-3*x^2+5*x))^(1/2)

Coefficient Matrix at [0 0]:
     0     1
    -1     0

Coefficient Matrix at [2 0]:
     0     1
    -3     0

Coefficient Matrix at [0.5 0]:
         0    1.0000
    0.7500         0

Eigenvalues at [0 0]:
   0.0000 + 1.0000i
  -0.0000 - 1.0000i

Eigenvalues at [2 0]:
   0.0000 + 1.7321i
  -0.0000 - 1.7321i

Eigenvalues at [0.5 0]:
    0.8660
   -0.8660

CRITICAL POINTS AND EIGENVALUES FOR b = -1
Critical Points:
     0     0
     1     0
     1     0

Coefficient Matrix:
[                                    0,                                    1]
[ -3/2*x^2+2*x-1+1/2*conj(-3*x^2+4*x),                                    0]

Eigenvalues:
  1/2*(-4-6*x^2+8*x+2*conj(-3*x^2+4*x))^(1/2)
 -1/2*(-4-6*x^2+8*x+2*conj(-3*x^2+4*x))^(1/2)

Coefficient Matrix at [0 0]:
     0     1
    -1     0

Coefficient Matrix at [1 0]:
     0     1
     0     0

Coefficient Matrix at [1 0]:
     0     1
     0     0

Eigenvalues at [0 0]:
   0.0000 + 1.0000i
  -0.0000 - 1.0000i

Eigenvalues at [1 0]:
     0
     0

Eigenvalues at [1 0]:
     0
     0

CRITICAL POINTS AND EIGENVALUES FOR b = -0.75
Critical Points:
        0                  0
   0.7500 + 0.6614i        0
   0.7500 - 0.6614i        0
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Coefficient Matrix:
[                                     0,                                      1]
[ -3/2*x^2+3/2*x-1+1/2*conj(-3*x^2+3*x),                                      0]

Eigenvalues:
  1/2*(-4-6*x^2+6*x+2*conj(-3*x^2+3*x))^(1/2)
 -1/2*(-4-6*x^2+6*x+2*conj(-3*x^2+3*x))^(1/2)

Coefficient Matrix at [0 0]:
     0     1
    -1     0

Coefficient Matrix at [0.75+0.66144i 0]:
         0    1.0000
    0.8750         0

Coefficient Matrix at [0.75-0.66144i 0]:
         0    1.0000
    0.8750         0

Eigenvalues at [0 0]:
   0.0000 + 1.0000i
  -0.0000 - 1.0000i

Eigenvalues at [0.75+0.66144i 0]:
    0.9354
   -0.9354

Eigenvalues at [0.75-0.66144i 0]:
    0.9354
   -0.9354

CRITICAL POINTS AND EIGENVALUES FOR b = -0.5
Critical Points:
        0                  0
   0.5000 + 0.8660i        0
   0.5000 - 0.8660i        0

Coefficient Matrix:
[                                 0,                                  1]
[ -3/2*x^2+x-1+1/2*conj(-3*x^2+2*x),                                  0]

Eigenvalues:
  1/2*(-4-6*x^2+4*x+2*conj(-3*x^2+2*x))^(1/2)
 -1/2*(-4-6*x^2+4*x+2*conj(-3*x^2+2*x))^(1/2)

Coefficient Matrix at [0 0]:
     0     1
    -1     0

Coefficient Matrix at [0.5+0.86603i 0]:
         0    1.0000
    1.5000         0

Coefficient Matrix at [0.5-0.86603i 0]:
         0    1.0000
    1.5000         0

Eigenvalues at [0 0]:
   0.0000 + 1.0000i
  -0.0000 - 1.0000i

Eigenvalues at [0.5+0.86603i 0]:
    1.2247
   -1.2247

Eigenvalues at [0.5-0.86603i 0]:
    1.2247
   -1.2247

CRITICAL POINTS AND EIGENVALUES FOR b = -0.25
Critical Points:
        0                  0
   0.2500 + 0.9682i        0
   0.2500 - 0.9682i        0

Coefficient Matrix:
[                                   0,                                    1]
[ -3/2*x^2+1/2*x-1+1/2*conj(-3*x^2+x),                                    0]
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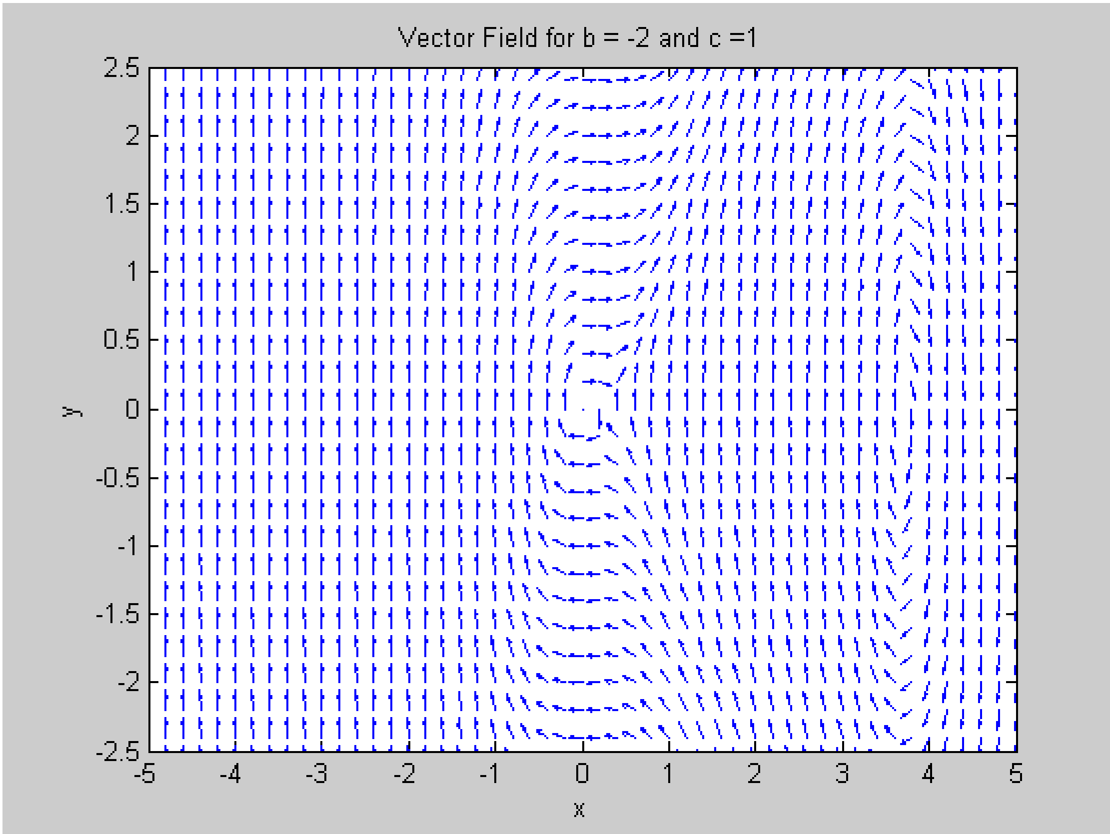
```
Eigenvalues:
  1/2*(-4-6*x^2+2*x+2*conj(-3*x^2+x))^(1/2)
 -1/2*(-4-6*x^2+2*x+2*conj(-3*x^2+x))^(1/2)

Coefficient Matrix at [0 0]:
     0     1
    -1     0

Coefficient Matrix at [0.25+0.96825i 0]:
         0    1.0000
    1.8750         0

Coefficient Matrix at [0.25-0.96825i 0]:
         0    1.0000
    1.8750         0

Eigenvalues at [0 0]:
   0.0000 + 1.0000i
  -0.0000 - 1.0000i

Eigenvalues at [0.25+0.96825i 0]:
    1.3693
   -1.3693

Eigenvalues at [0.25-0.96825i 0]:
    1.3693
   -1.3693

CRITICAL POINTS AND EIGENVALUES FOR b = 0
Critical Points:
        0                  0
   0.0000 + 1.0000i        0
  -0.0000 - 1.0000i        0

Coefficient Matrix:
[                        0,                          1]
[ -3/2*x^2-1-3/2*conj(x)^2,                          0]

Eigenvalues:
  1/2*(-4-6*x^2-6*conj(x)^2)^(1/2)
 -1/2*(-4-6*x^2-6*conj(x)^2)^(1/2)

Coefficient Matrix at [0 0]:
     0     1
    -1     0

Coefficient Matrix at [6.1232e-017+1i 0]:
     0     1
     2     0

Coefficient Matrix at [-6.1232e-017-1i 0]:
     0     1
     2     0

Eigenvalues at [0 0]:
   0.0000 + 1.0000i
  -0.0000 - 1.0000i

Eigenvalues at [6.1232e-017+1i 0]:
    1.4142
   -1.4142

Eigenvalues at [-6.1232e-017-1i 0]:
    1.4142
   -1.4142

CRITICAL POINTS AND EIGENVALUES FOR b = 0.25
Critical Points:
        0                  0
  -0.2500 + 0.9682i        0
  -0.2500 - 0.9682i        0

Coefficient Matrix:
[                                   0,                                     1]
[ -3/2*x^2-1/2*x-1+1/2*conj(-3*x^2-x),                                     0]

Eigenvalues:
  1/2*(-4-6*x^2-2*x+2*conj(-3*x^2-x))^(1/2)
 -1/2*(-4-6*x^2-2*x+2*conj(-3*x^2-x))^(1/2)

Coefficient Matrix at [0 0]:
```

```
     0     1
    -1     0

Coefficient Matrix at [-0.25+0.96825i 0]:
         0    1.0000
    1.8750         0

Coefficient Matrix at [-0.25-0.96825i 0]:
         0    1.0000
    1.8750         0

Eigenvalues at [0 0]:
   0.0000 + 1.0000i
  -0.0000 - 1.0000i

Eigenvalues at [-0.25+0.96825i 0]:
    1.3693
   -1.3693

Eigenvalues at [-0.25-0.96825i 0]:
    1.3693
   -1.3693

CRITICAL POINTS AND EIGENVALUES FOR b = 0.5
Critical Points:
        0                  0
  -0.5000 + 0.8660i        0
  -0.5000 - 0.8660i        0

Coefficient Matrix:
[                              0,                                  1]
[ -3/2*x^2-x-1+1/2*conj(-3*x^2-2*x),                               0]

Eigenvalues:
  1/2*(-4-6*x^2-4*x+2*conj(-3*x^2-2*x))^(1/2)
 -1/2*(-4-6*x^2-4*x+2*conj(-3*x^2-2*x))^(1/2)

Coefficient Matrix at [0 0]:
     0     1
    -1     0

Coefficient Matrix at [-0.5+0.86603i 0]:
         0    1.0000
    1.5000         0

Coefficient Matrix at [-0.5-0.86603i 0]:
         0    1.0000
    1.5000         0

Eigenvalues at [0 0]:
   0.0000 + 1.0000i
  -0.0000 - 1.0000i

Eigenvalues at [-0.5+0.86603i 0]:
    1.2247
   -1.2247

Eigenvalues at [-0.5-0.86603i 0]:
    1.2247
   -1.2247

CRITICAL POINTS AND EIGENVALUES FOR b = 0.75
Critical Points:
        0                  0
  -0.7500 + 0.6614i        0
  -0.7500 - 0.6614i        0

Coefficient Matrix:
[                                  0,                                  1]
[ -3/2*x^2-3/2*x-1+1/2*conj(-3*x^2-3*x),                               0]

Eigenvalues:
  1/2*(-4-6*x^2-6*x+2*conj(-3*x^2-3*x))^(1/2)
 -1/2*(-4-6*x^2-6*x+2*conj(-3*x^2-3*x))^(1/2)

Coefficient Matrix at [0 0]:
     0     1
    -1     0

Coefficient Matrix at [-0.75+0.66144i 0]:
         0    1.0000
```

```
    0.8750         0

Coefficient Matrix at [-0.75-0.66144i 0]:
         0    1.0000
    0.8750         0

Eigenvalues at [0 0]:
   0.0000 + 1.0000i
  -0.0000 - 1.0000i

Eigenvalues at [-0.75+0.66144i 0]:
    0.9354
   -0.9354

Eigenvalues at [-0.75-0.66144i 0]:
    0.9354
   -0.9354

CRITICAL POINTS AND EIGENVALUES FOR b = 1
Critical Points:
     0     0
    -1     0
    -1     0

Coefficient Matrix:
[                                    0,                                    1]
[ -3/2*x^2-2*x-1+1/2*conj(-3*x^2-4*x),                                    0]

Eigenvalues:
  1/2*(-4-6*x^2-8*x+2*conj(-3*x^2-4*x))^(1/2)
 -1/2*(-4-6*x^2-8*x+2*conj(-3*x^2-4*x))^(1/2)

Coefficient Matrix at [0 0]:
     0     1
    -1     0

Coefficient Matrix at [-1 0]:
     0     1
     0     0

Coefficient Matrix at [-1 0]:
     0     1
     0     0

Eigenvalues at [0 0]:
   0.0000 + 1.0000i
  -0.0000 - 1.0000i

Eigenvalues at [-1 0]:
     0
     0

Eigenvalues at [-1 0]:
     0
     0

CRITICAL POINTS AND EIGENVALUES FOR b = 1.25
Critical Points:
         0         0
   -0.5000         0
   -2.0000         0

Coefficient Matrix:
[                                      0,                                      1]
[ -3/2*x^2-5/2*x-1+1/2*conj(-3*x^2-5*x),                                      0]

Eigenvalues:
  1/2*(-4-6*x^2-10*x+2*conj(-3*x^2-5*x))^(1/2)
 -1/2*(-4-6*x^2-10*x+2*conj(-3*x^2-5*x))^(1/2)

Coefficient Matrix at [0 0]:
     0     1
    -1     0

Coefficient Matrix at [-0.5 0]:
         0    1.0000
    0.7500         0

Coefficient Matrix at [-2 0]:
     0     1
    -3     0
```

```
Eigenvalues at [0 0]:
   0.0000 + 1.0000i
  -0.0000 - 1.0000i

Eigenvalues at [-0.5 0]:
    0.8660
   -0.8660

Eigenvalues at [-2 0]:
   0.0000 + 1.7321i
  -0.0000 - 1.7321i

CRITICAL POINTS AND EIGENVALUES FOR b = 1.5
Critical Points:
         0         0
   -0.3820         0
   -2.6180         0

Coefficient Matrix:
[                                    0,                                    1]
[ -3/2*x^2-3*x-1+1/2*conj(-3*x^2-6*x),                                    0]

Eigenvalues:
  1/2*(-4-6*x^2-12*x+2*conj(-3*x^2-6*x))^(1/2)
 -1/2*(-4-6*x^2-12*x+2*conj(-3*x^2-6*x))^(1/2)

Coefficient Matrix at [0 0]:
     0     1
    -1     0

Coefficient Matrix at [-0.38197 0]:
         0    1.0000
    0.8541         0

Coefficient Matrix at [-2.618 0]:
         0    1.0000
   -5.8541         0

Eigenvalues at [0 0]:
   0.0000 + 1.0000i
  -0.0000 - 1.0000i

Eigenvalues at [-0.38197 0]:
    0.9242
   -0.9242

Eigenvalues at [-2.618 0]:
   0.0000 + 2.4195i
  -0.0000 - 2.4195i

CRITICAL POINTS AND EIGENVALUES FOR b = 1.75
Critical Points:
         0         0
   -0.3139         0
   -3.1861         0

Coefficient Matrix:
[                                      0,                                      1]
[ -3/2*x^2-7/2*x-1+1/2*conj(-3*x^2-7*x),                                      0]

Eigenvalues:
  1/2*(-4-6*x^2-14*x+2*conj(-3*x^2-7*x))^(1/2)
 -1/2*(-4-6*x^2-14*x+2*conj(-3*x^2-7*x))^(1/2)

Coefficient Matrix at [0 0]:
     0     1
    -1     0

Coefficient Matrix at [-0.31386 0]:
         0    1.0000
    0.9015         0

Coefficient Matrix at [-3.1861 0]:
         0    1.0000
   -9.1515         0

Eigenvalues at [0 0]:
   0.0000 + 1.0000i
  -0.0000 - 1.0000i
```

```
Eigenvalues at [-0.31386 0]:
    0.9495
   -0.9495

Eigenvalues at [-3.1861 0]:
   0.0000 + 3.0251i
  -0.0000 - 3.0251i

CRITICAL POINTS AND EIGENVALUES FOR b = 2
Critical Points:
         0         0
   -0.2679         0
   -3.7321         0

Coefficient Matrix:
[                                   0,                                    1]
[ -3/2*x^2-4*x-1+1/2*conj(-3*x^2-8*x),                                    0]

Eigenvalues:
  1/2*(-4-6*x^2-16*x+2*conj(-3*x^2-8*x))^(1/2)
 -1/2*(-4-6*x^2-16*x+2*conj(-3*x^2-8*x))^(1/2)

Coefficient Matrix at [0 0]:
     0     1
    -1     0

Coefficient Matrix at [-0.26795 0]:
         0    1.0000
    0.9282         0

Coefficient Matrix at [-3.7321 0]:
         0    1.0000
  -12.9282         0

Eigenvalues at [0 0]:
   0.0000 + 1.0000i
  -0.0000 - 1.0000i

Eigenvalues at [-0.26795 0]:
    0.9634
   -0.9634

Eigenvalues at [-3.7321 0]:
   0.0000 + 3.5956i
  -0.0000 - 3.5956i
```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-2
-2.5
-5
-4
-3
-2
-1
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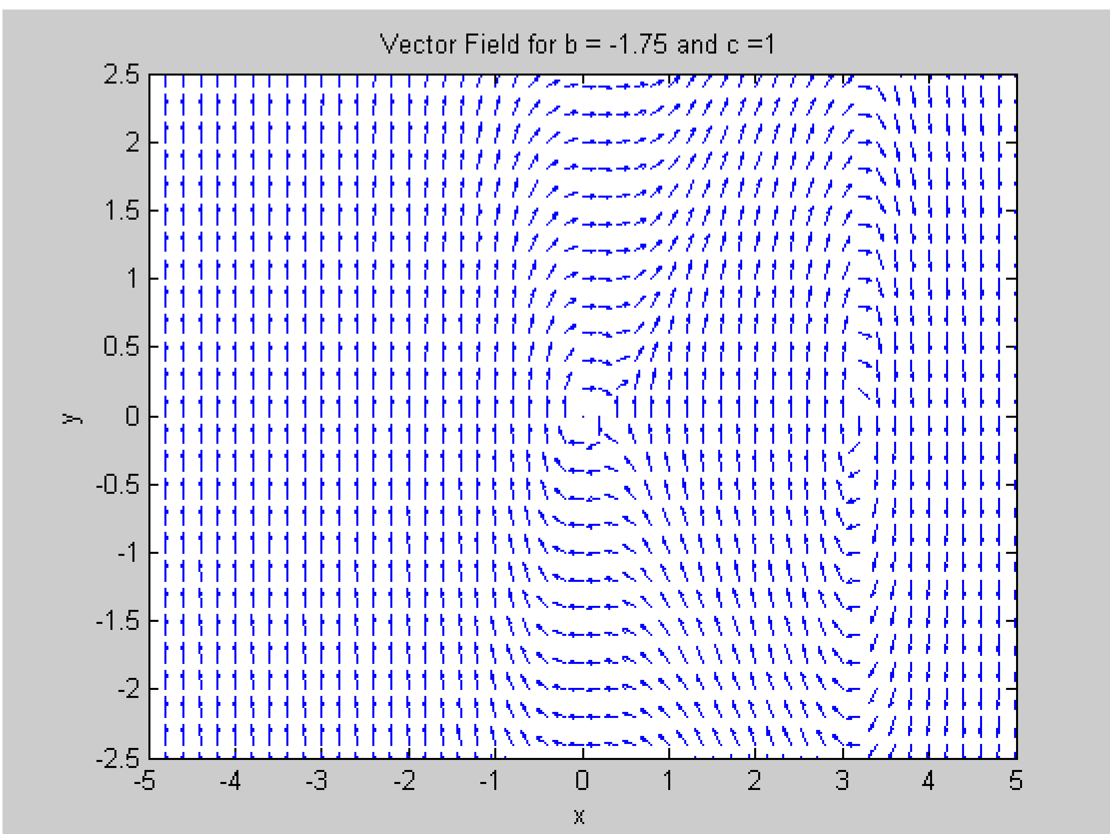
1
2
3
4
5


Vector Field for b = -1.75 and c =1
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x
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1.5
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0.5
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-0.5
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-2
-2.5
-5
-4
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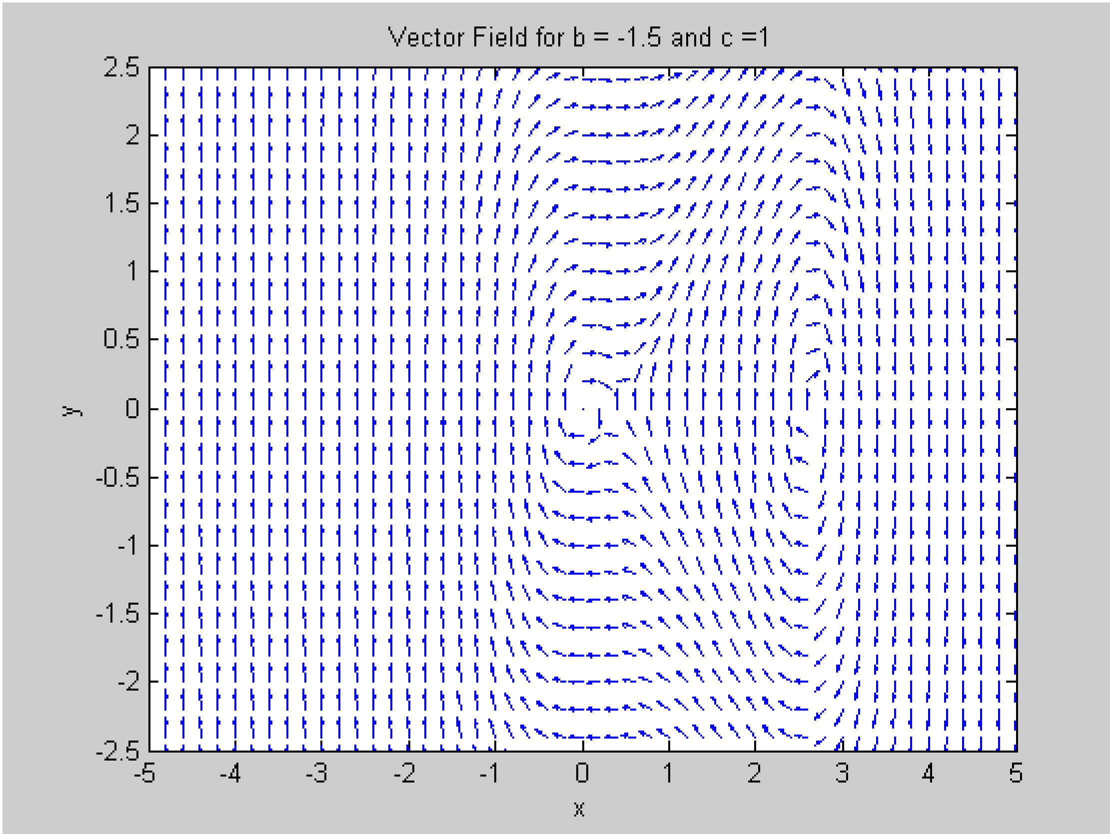
0
1
2
3
4
5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x
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1.5
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0.5
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-0.5
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-1.5
-2
-2.5
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-4
-3
-2
-1
0
1
2
3
4
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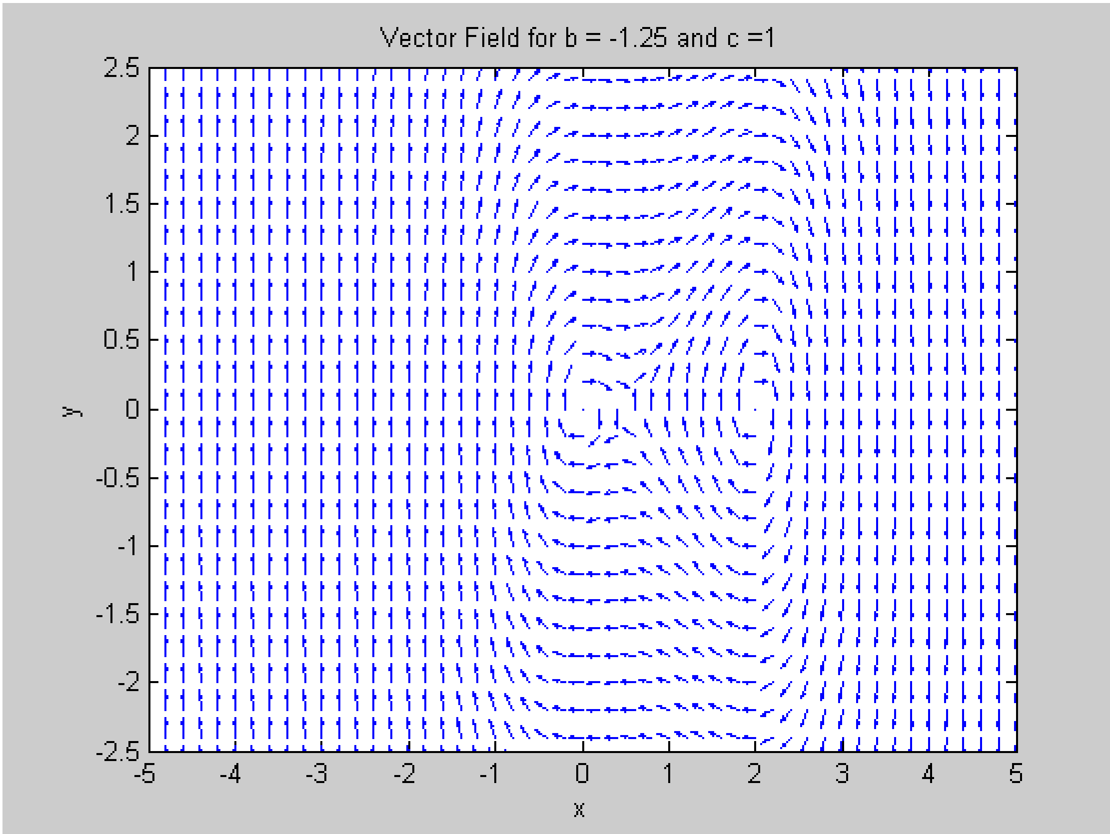

Vector Field for b = -1.25 and c =1
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1.5
1
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0
-0.5
-1
-1.5
-2
-2.5
y
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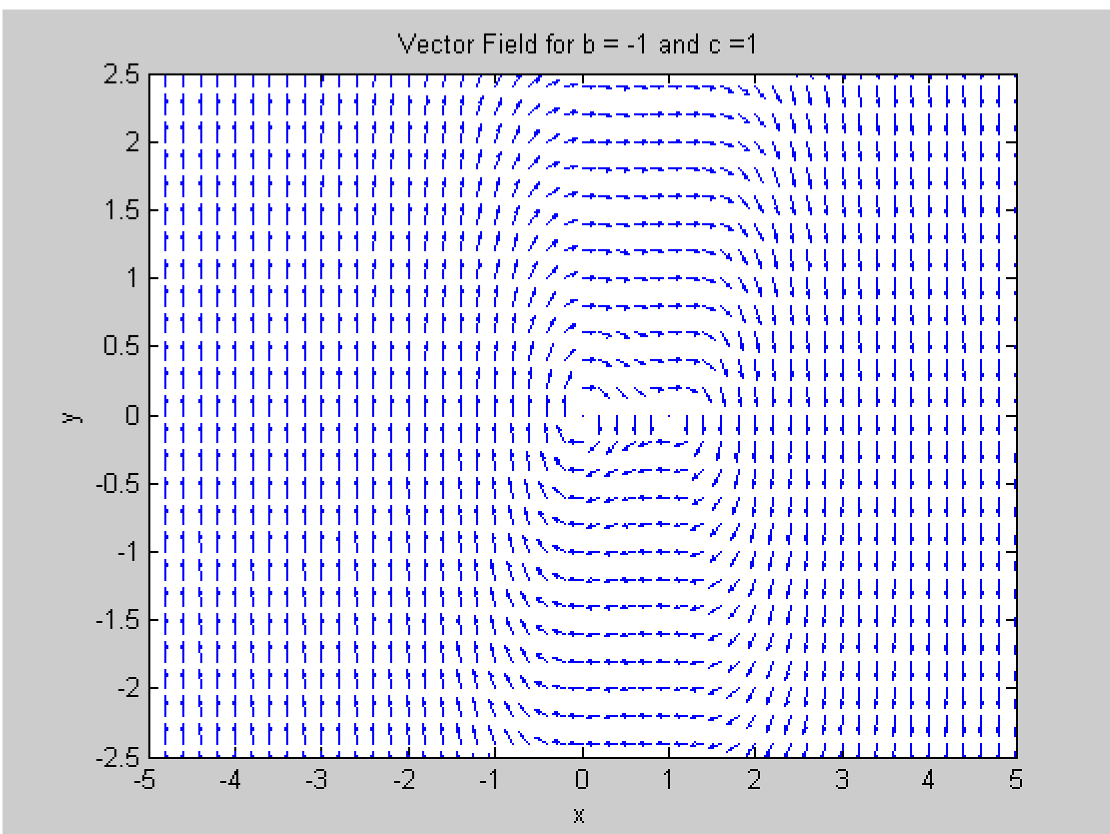
-4
-3
-2
-1
0
1
2
3
4
5
x



Vector Field for b = -1 and c =1
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1.5
1
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0
-0.5
-1
-1.5
-2
-2.5
y
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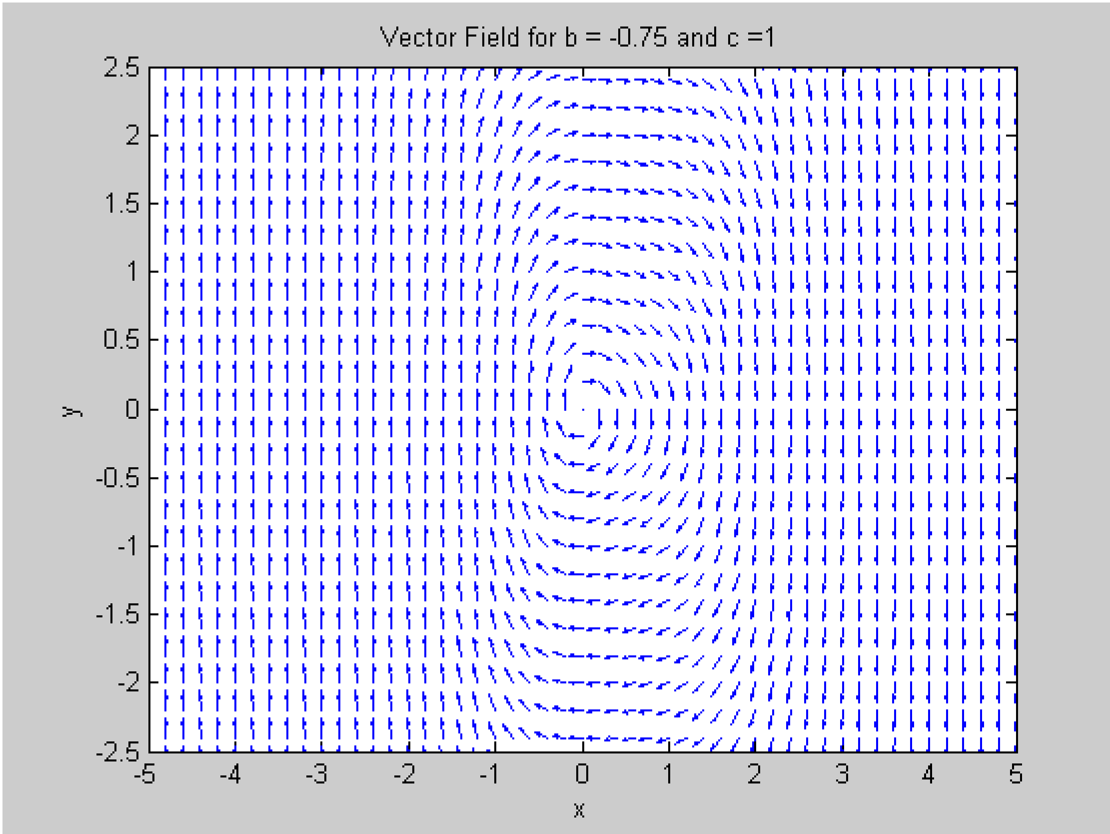
-4
-3
-2
-1
0
1
2
3
4
5
x


Vector Field for b = -0.75 and c =1
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1
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-1
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y
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-4
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1
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3
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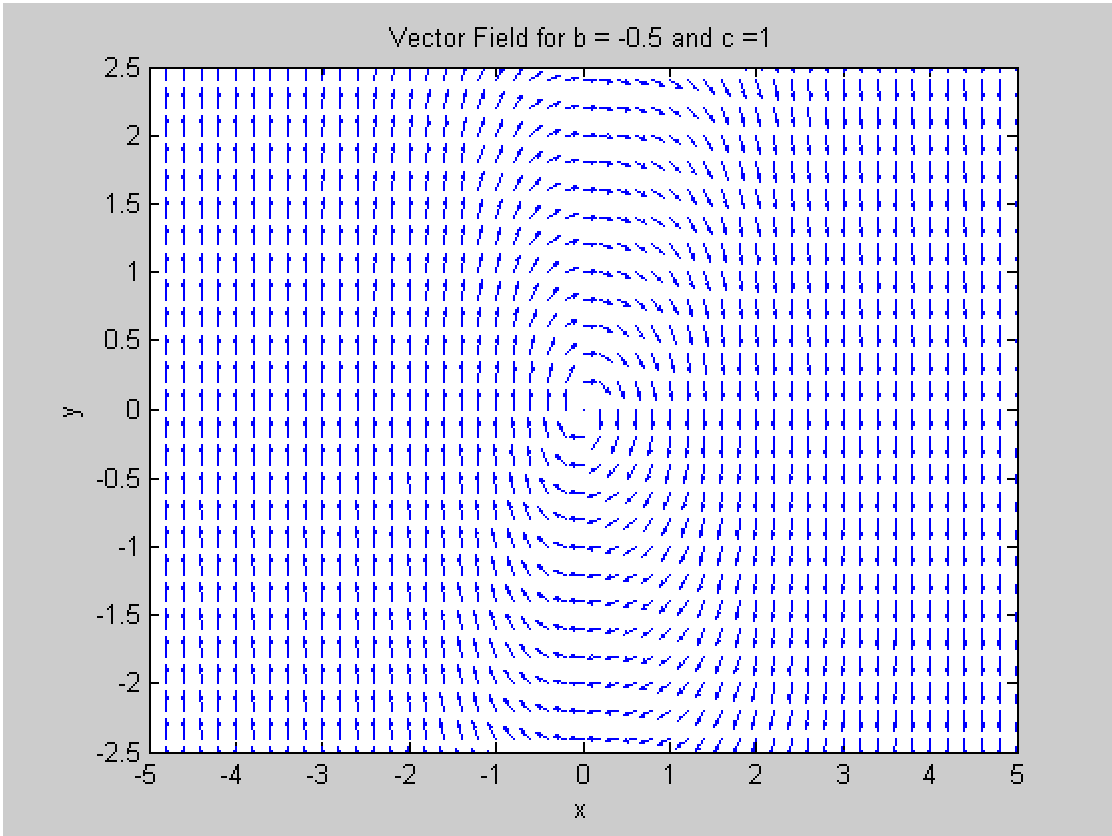
5
x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x
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1.5
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0.5
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-2
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-4
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-1
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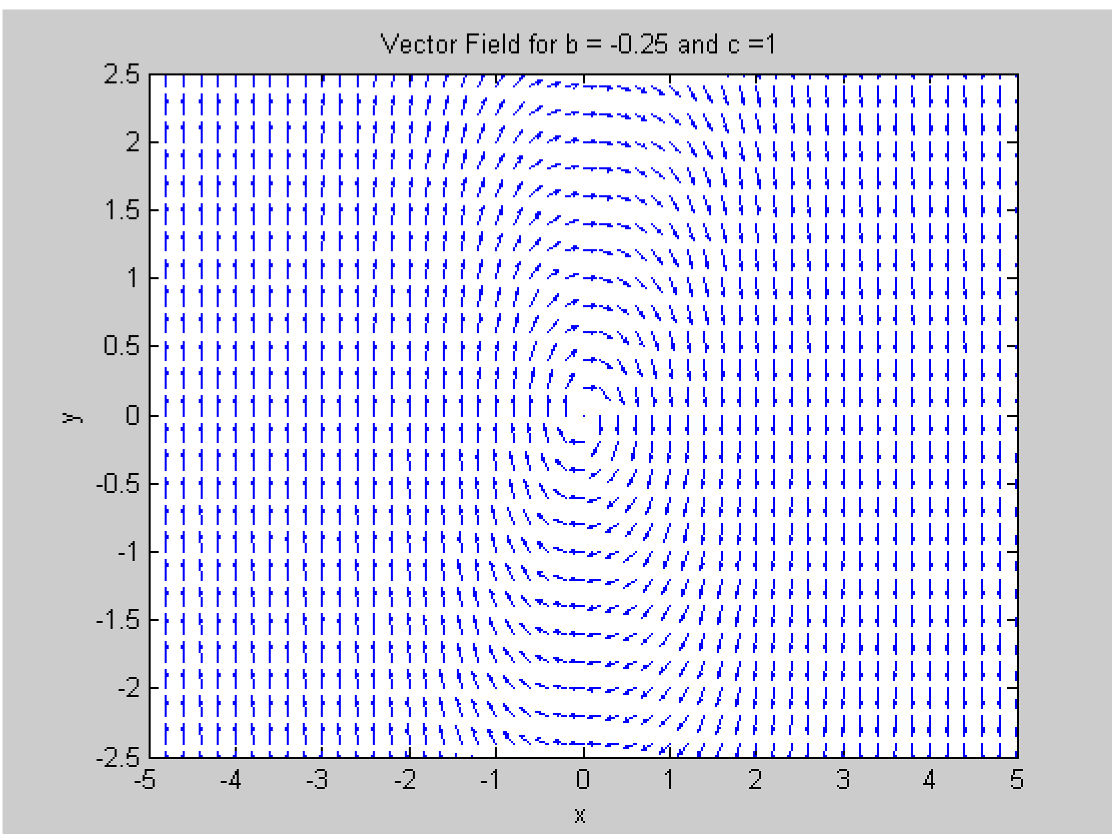
1
2
3
4
5


Vector Field for b = -0.25 and c =1
y
x
2.5
2
1.5
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0.5
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-0.5
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-2
-2.5
-5
-4
-3
-2
-1
0
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2
3
4
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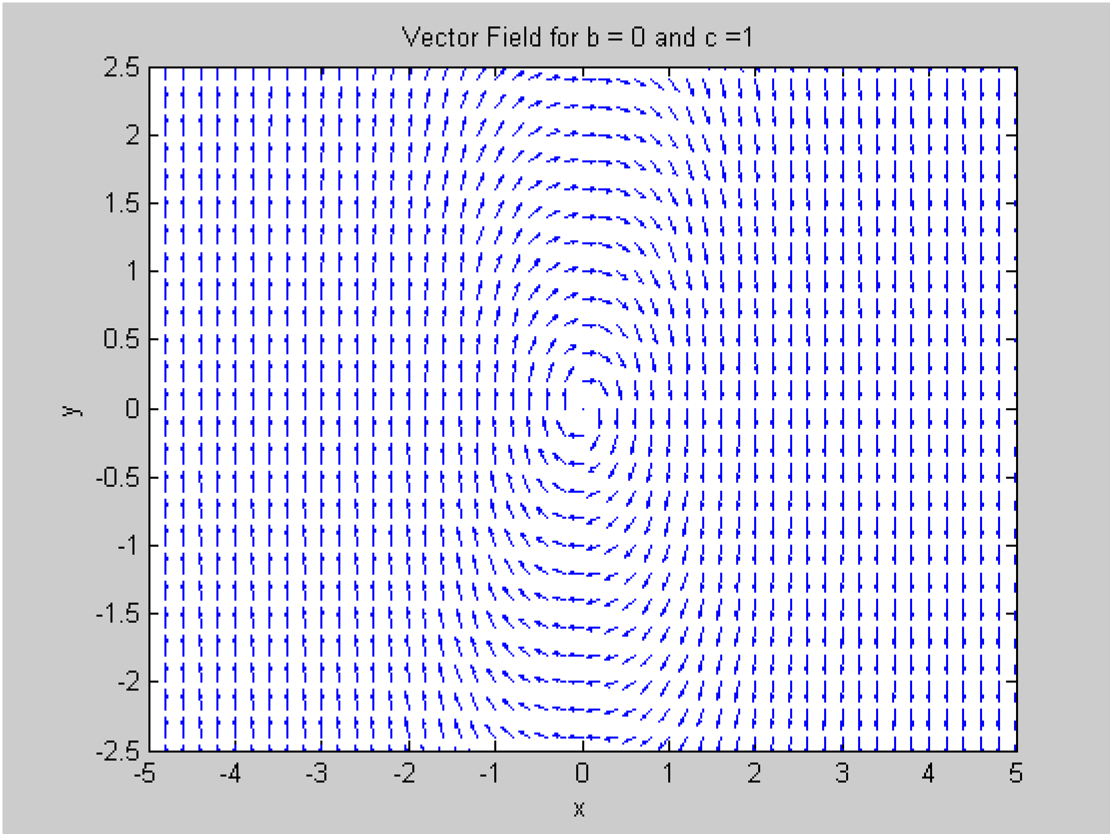
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1.5
1
0.5
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-0.5
-1
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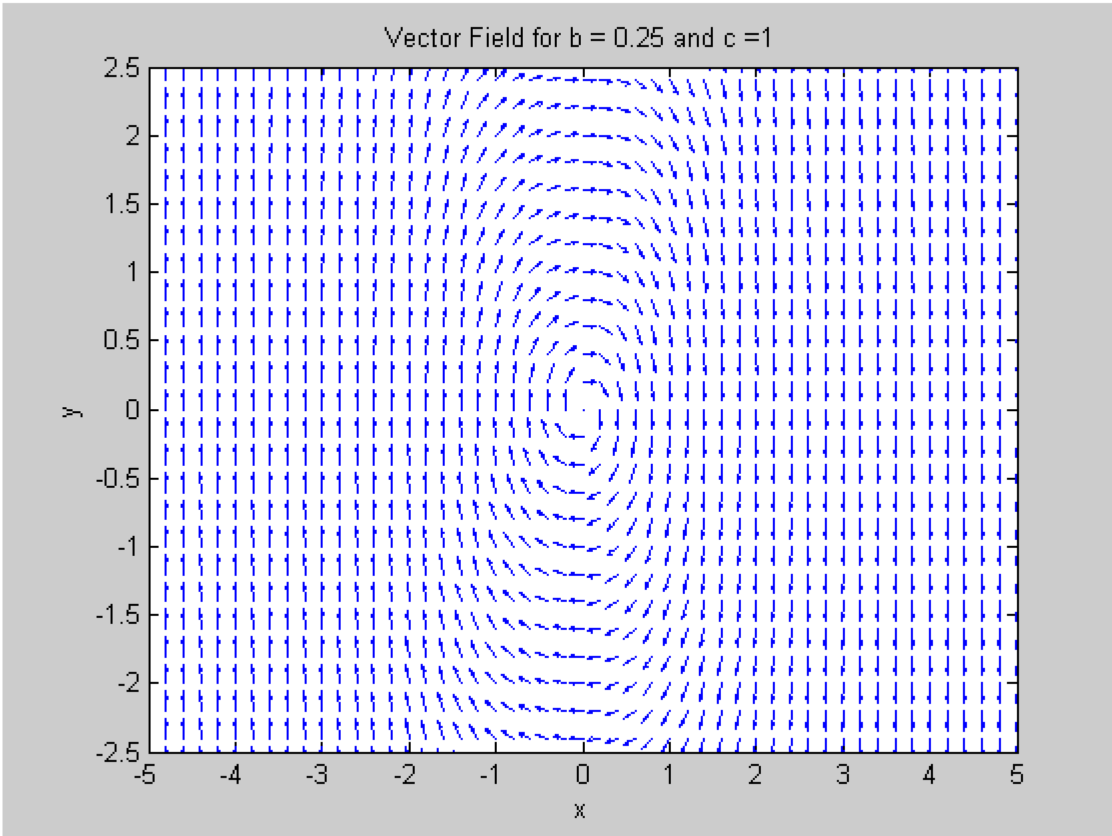
-2
-2.5
y
-5
-4
-3
-2
-1
0
1
2
3
4
5
x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x
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1.5
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0.5
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-0.5
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-2
-2.5
-5
-4
-3
-2
-1
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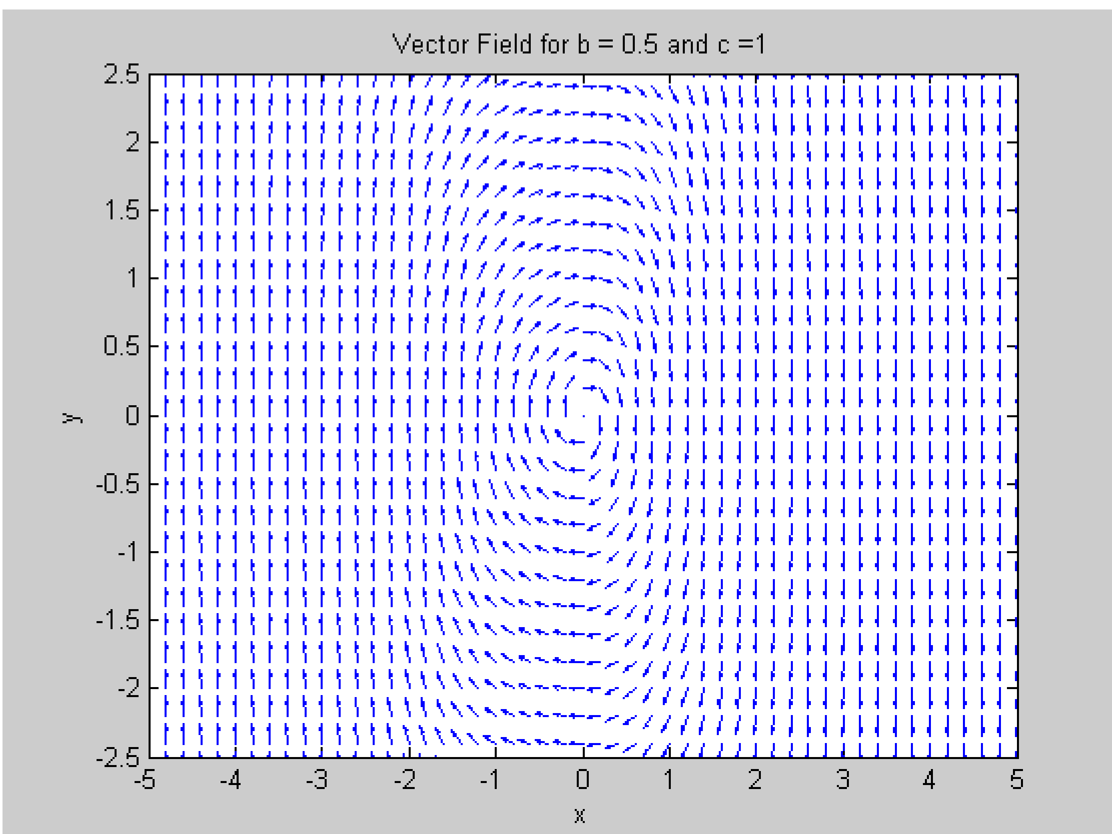
2
3
4
5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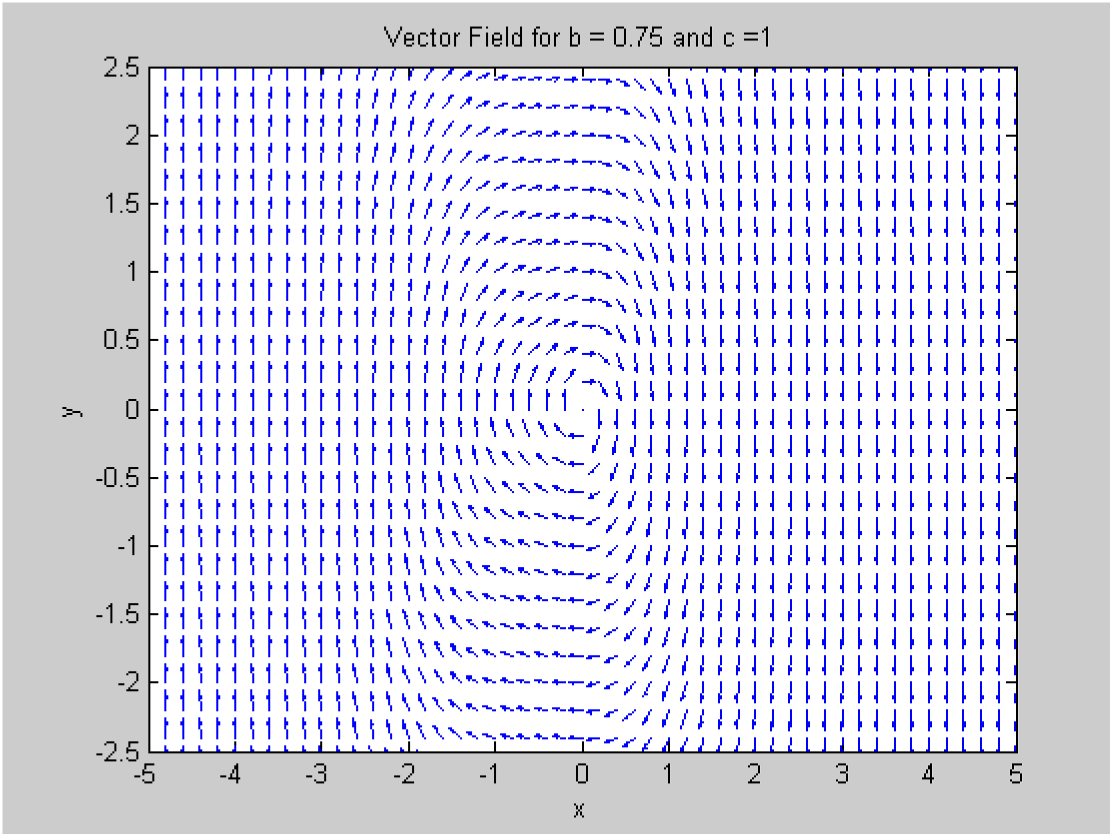

Vector Field for b = 0.75 and c =1
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0.5
0
-0.5
-1
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y
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0
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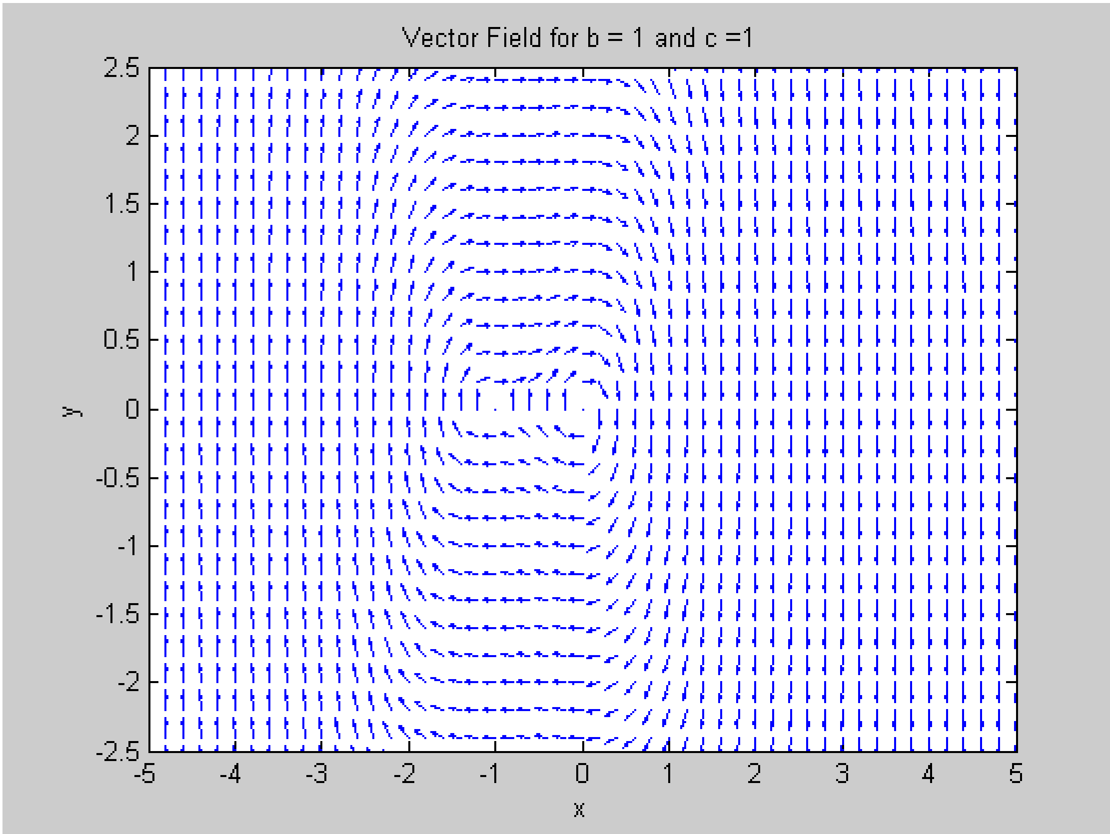
x


Vector Field for b = 1 and c =1
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-0.5
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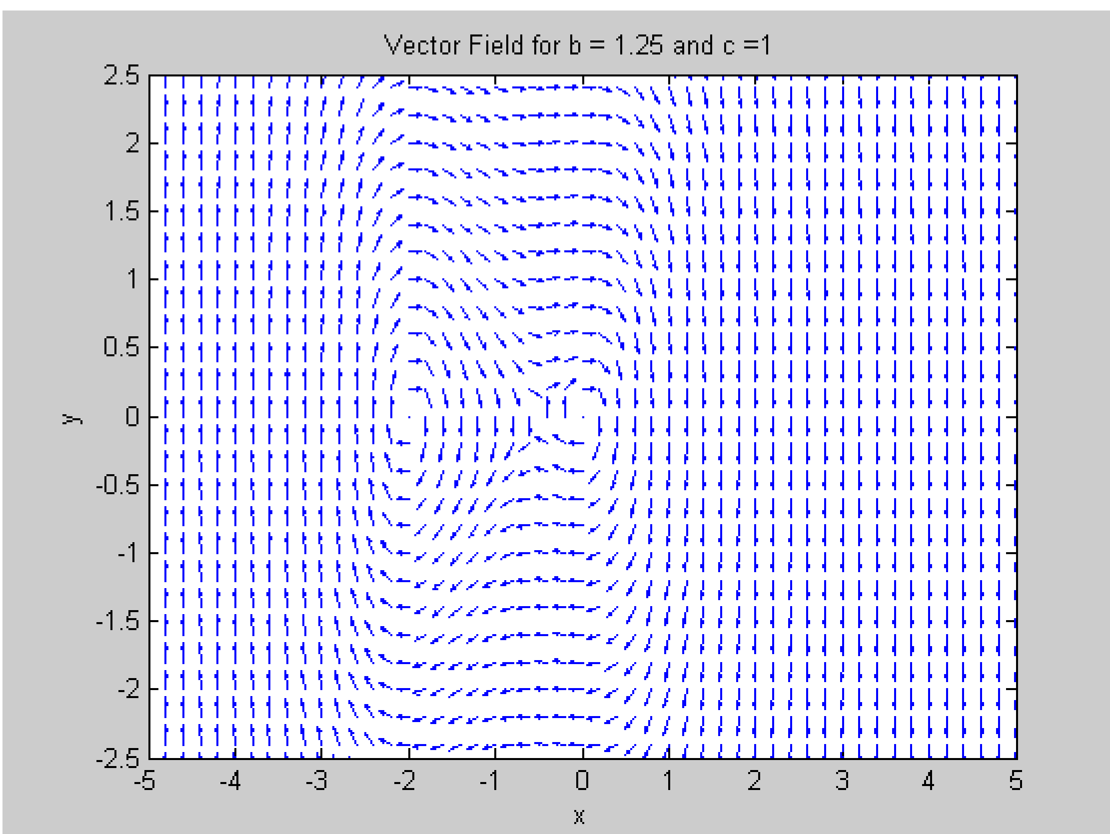
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y
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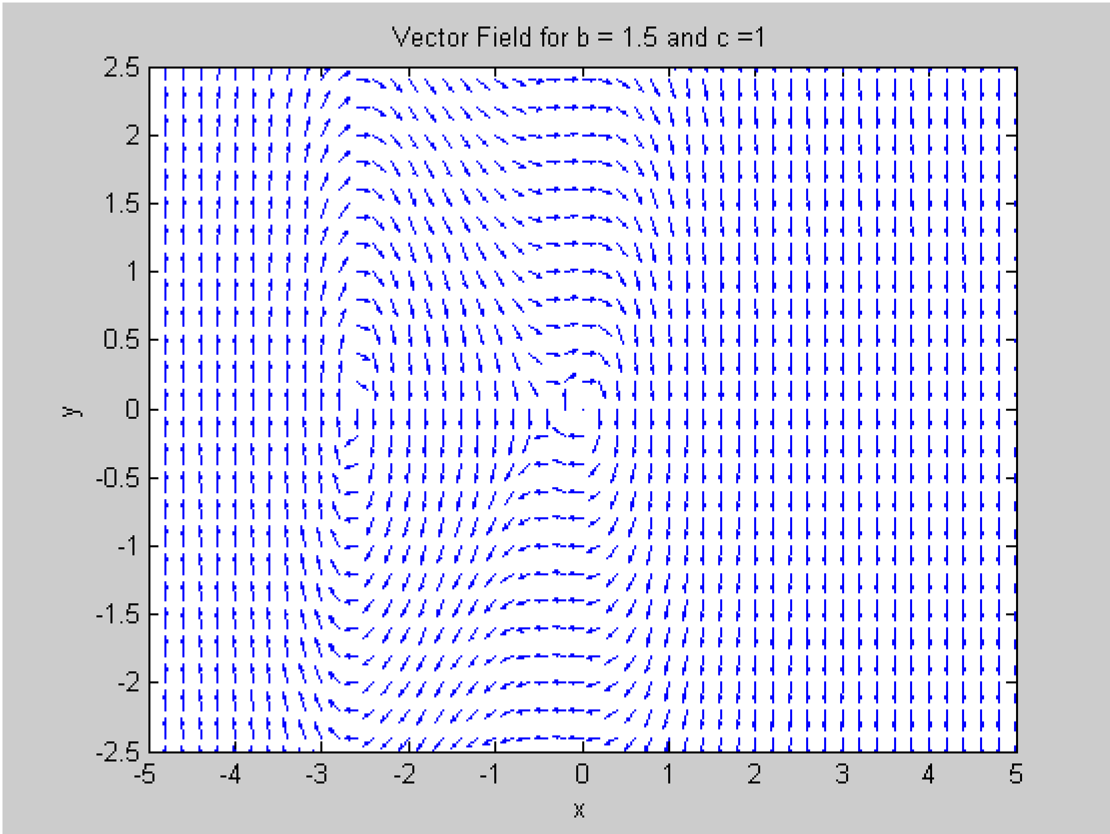

Vector Field for b = 1.5 and c =1
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y
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-4
-3
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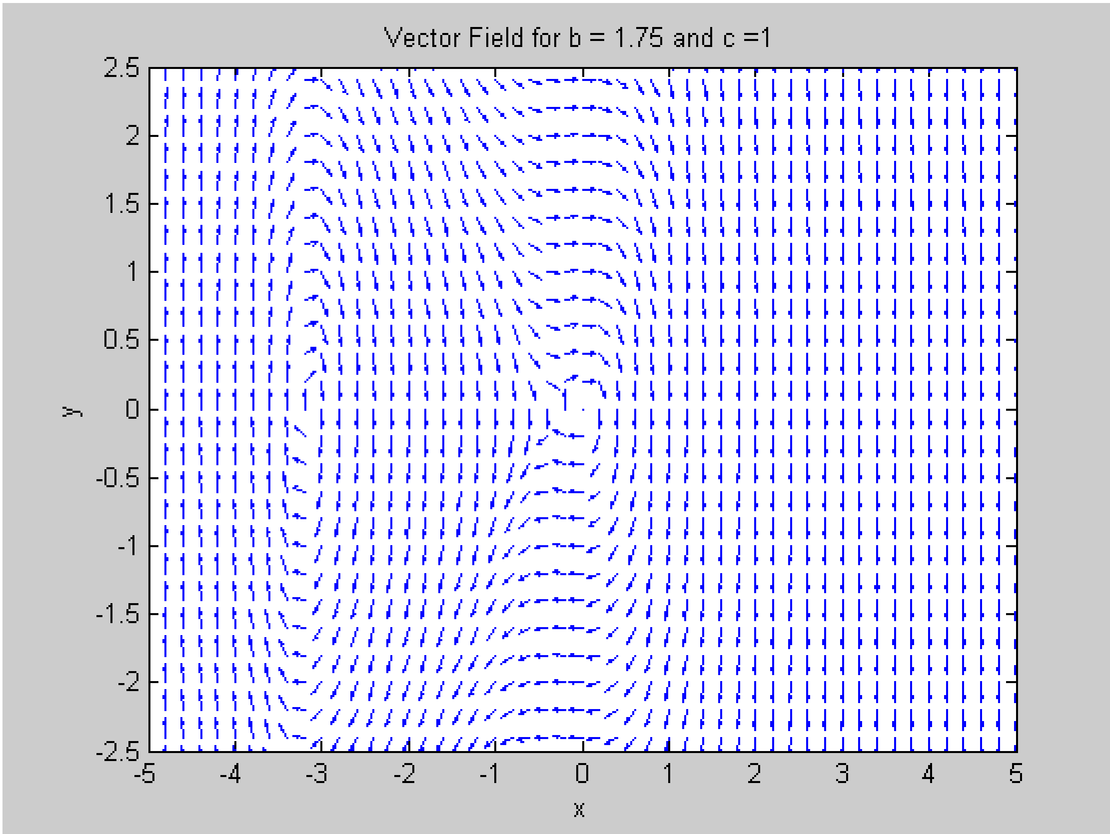
-1
0
1
2
3
4
5
x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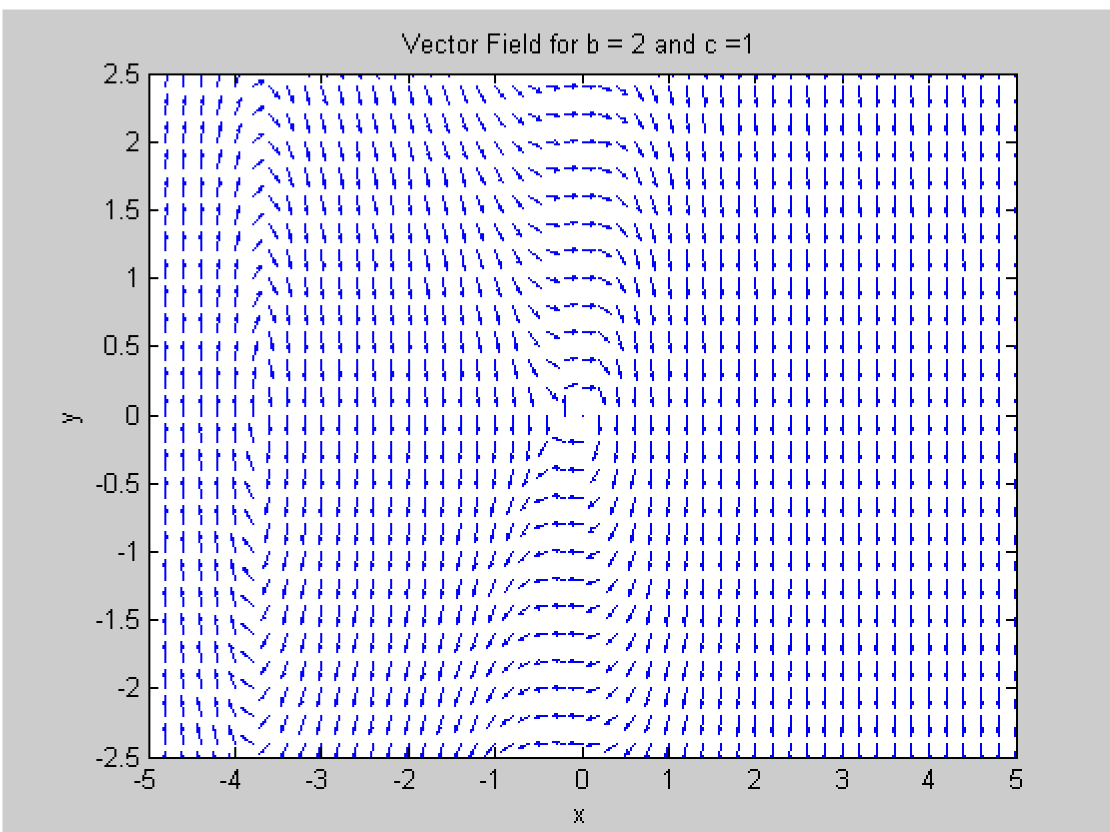
x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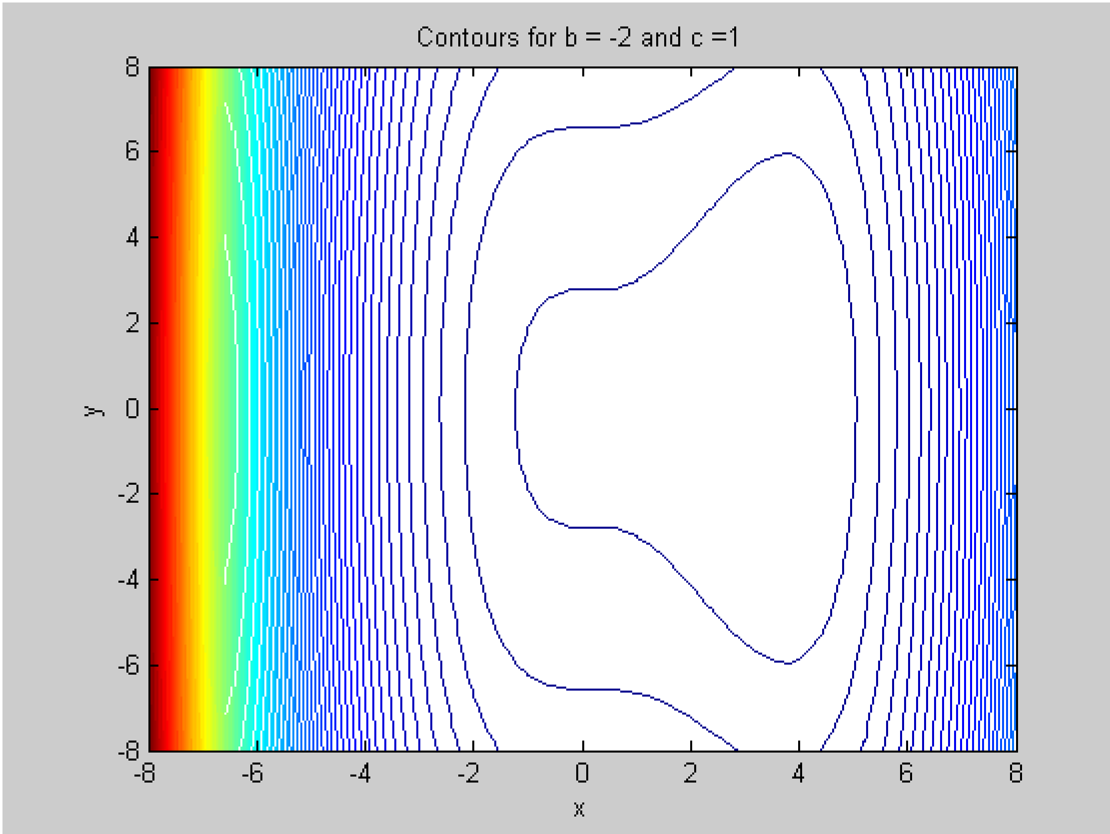
Contours for b = -2 and c =1
y
x
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6
4
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-4
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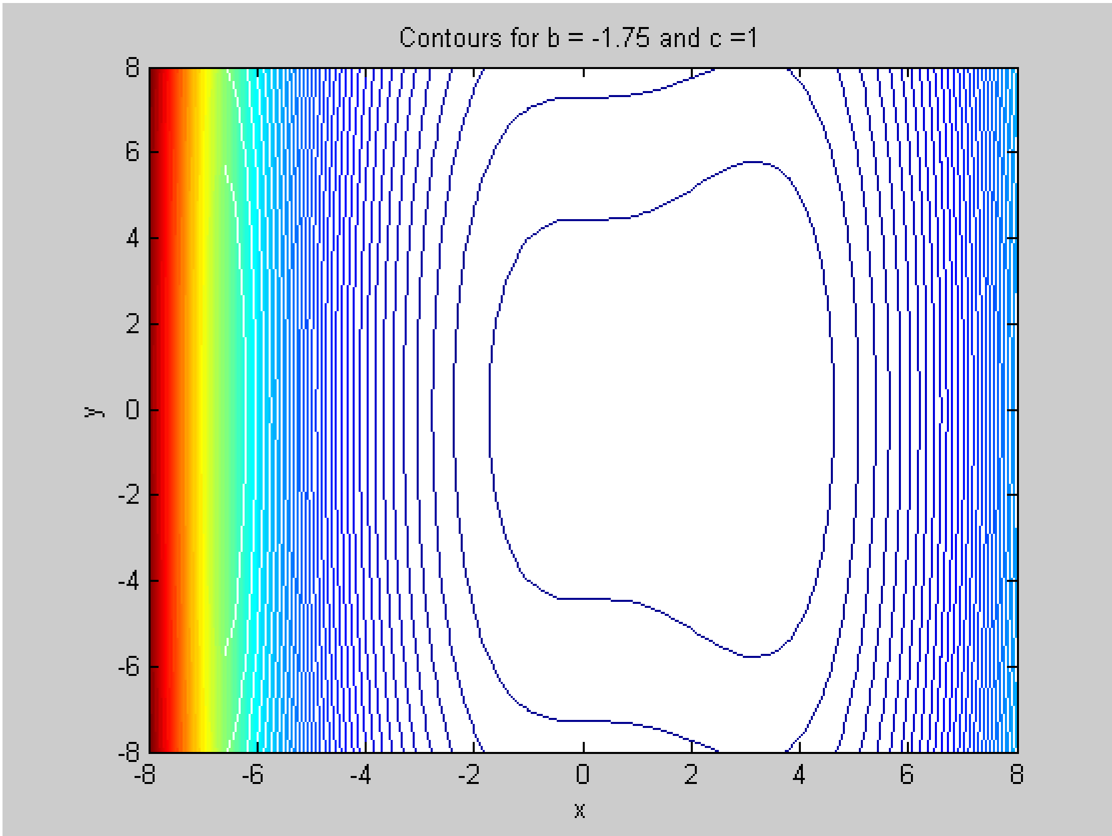
8


Contours for b = -1.75 and c =1
y
x
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4
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-4
-6
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-8
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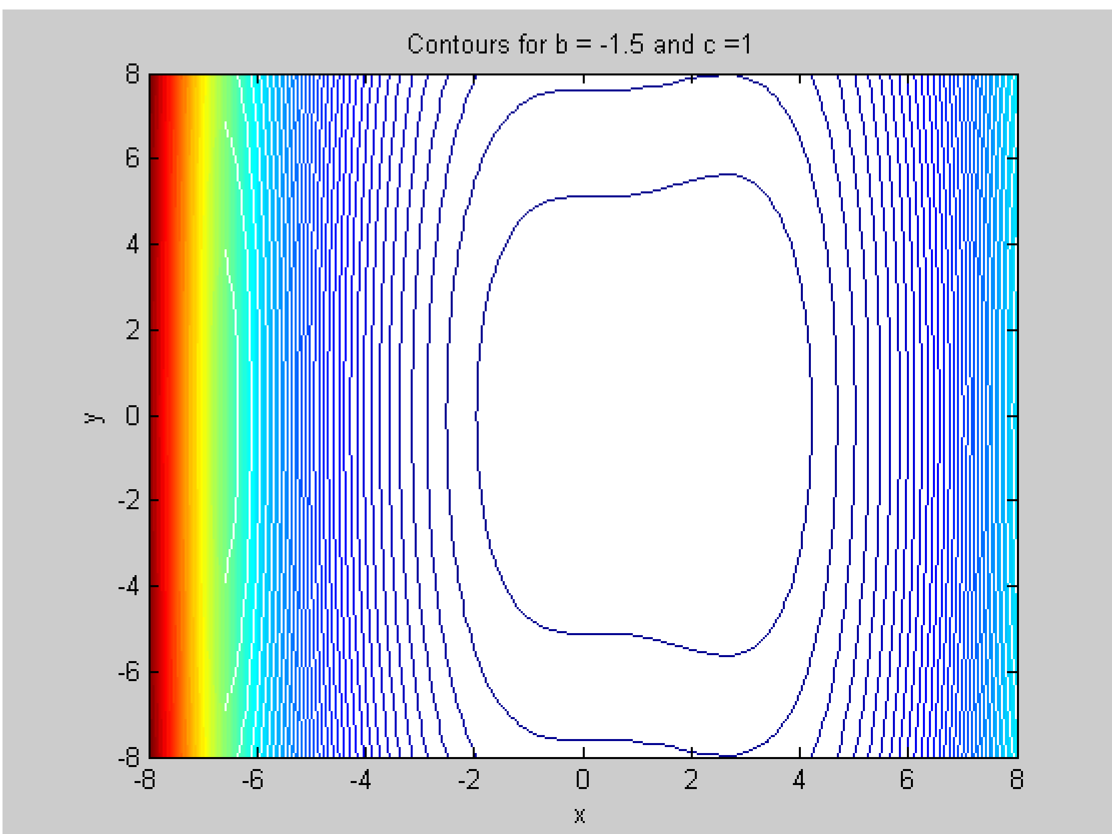
Contours for b = -1.5 and c =1
y
x
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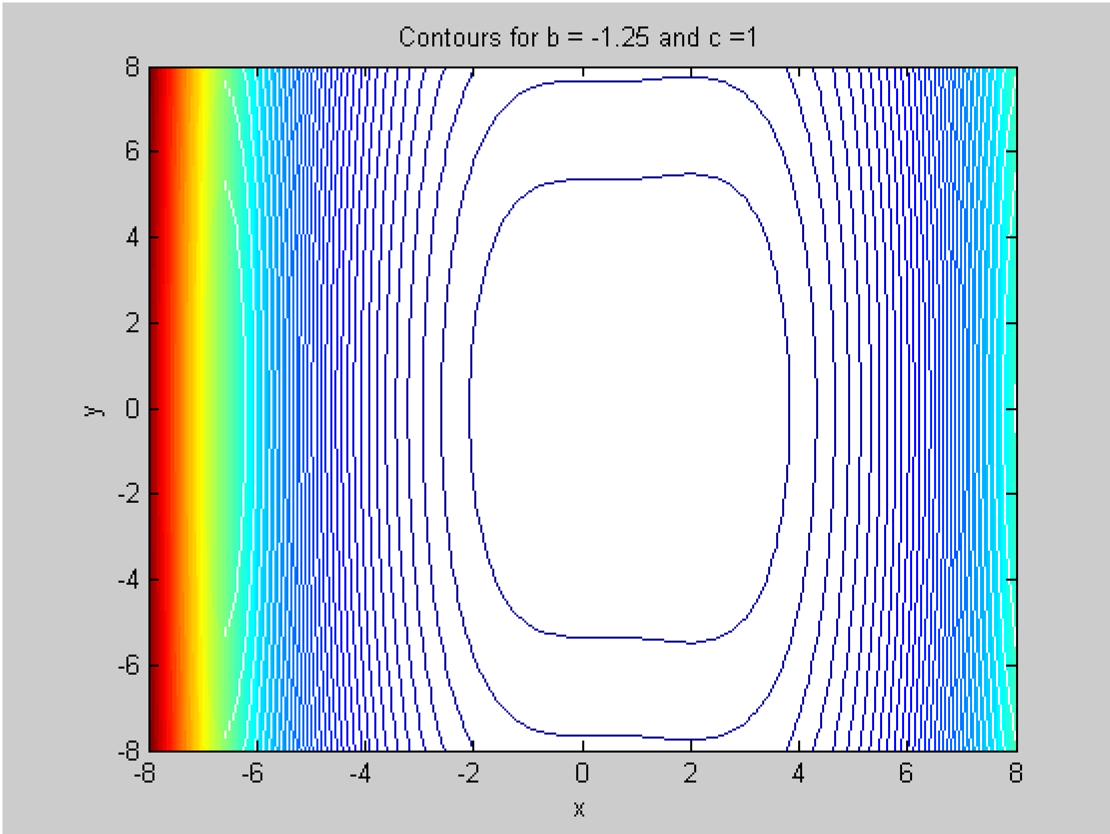
Contours for b = -1.25 and c =1
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x
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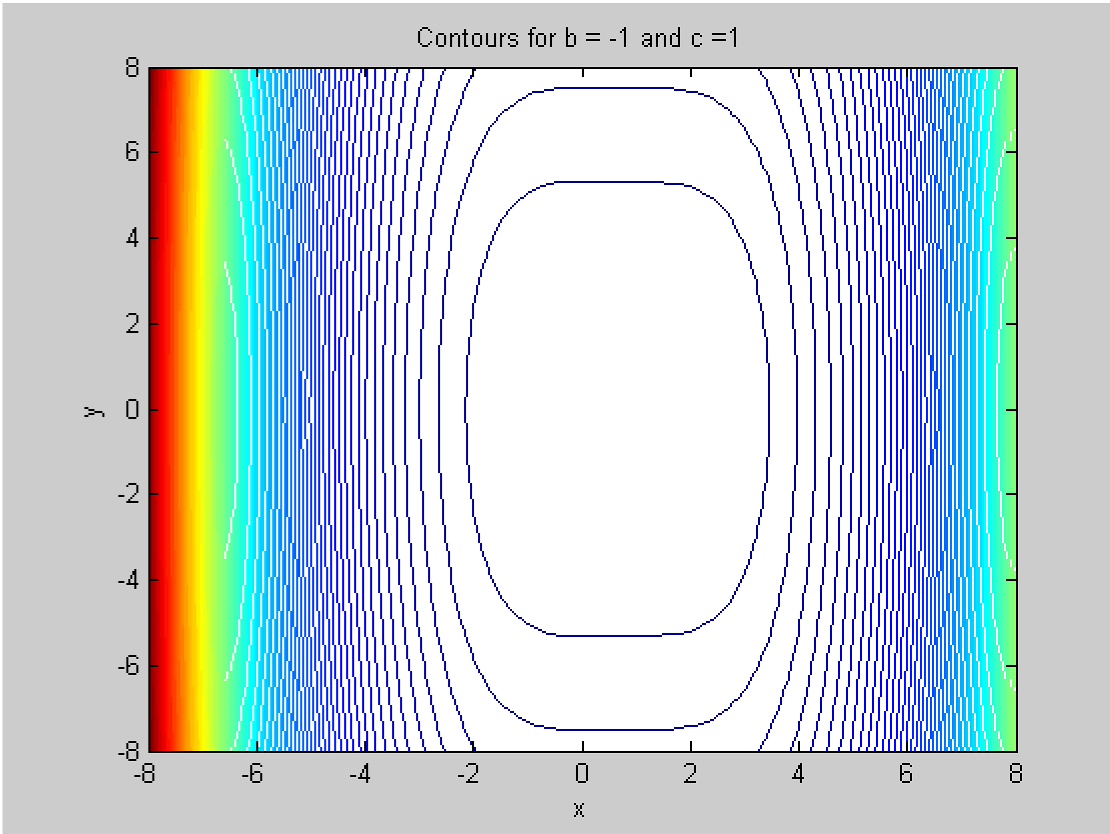
4
6
8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x
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2
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-4
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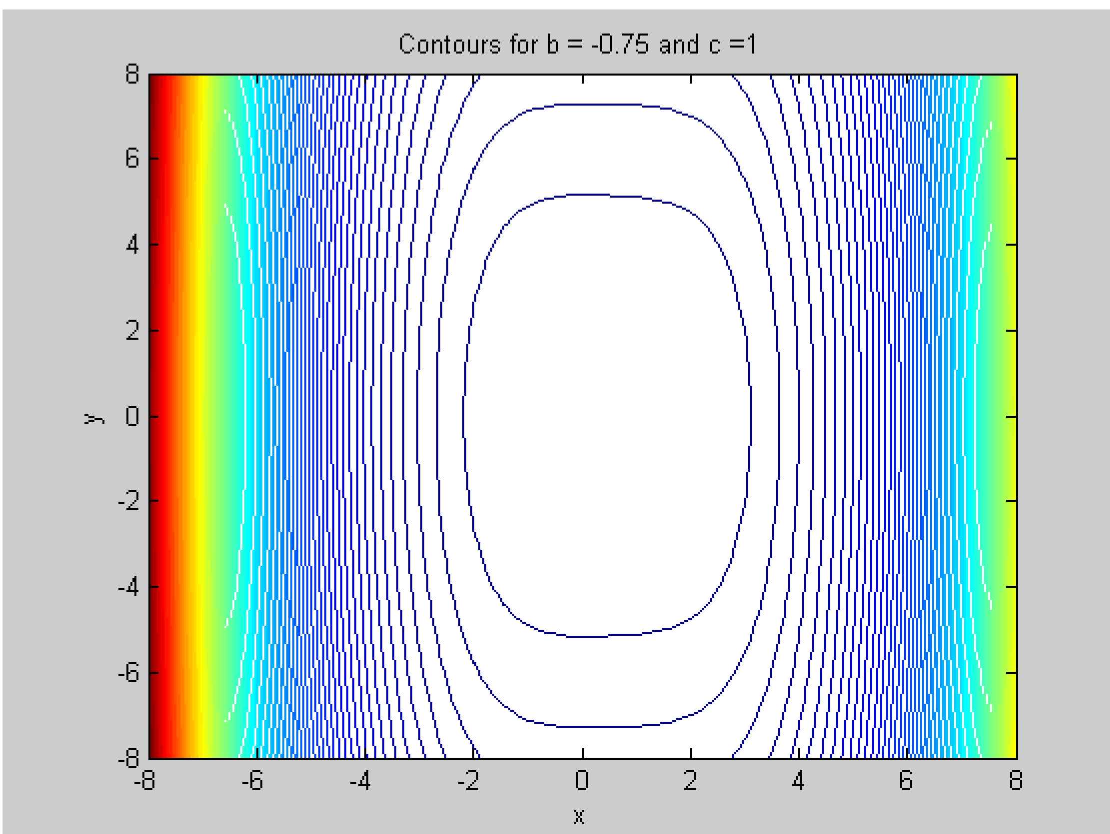
-8


Contours for b = -0.75 and c =1
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x
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-4
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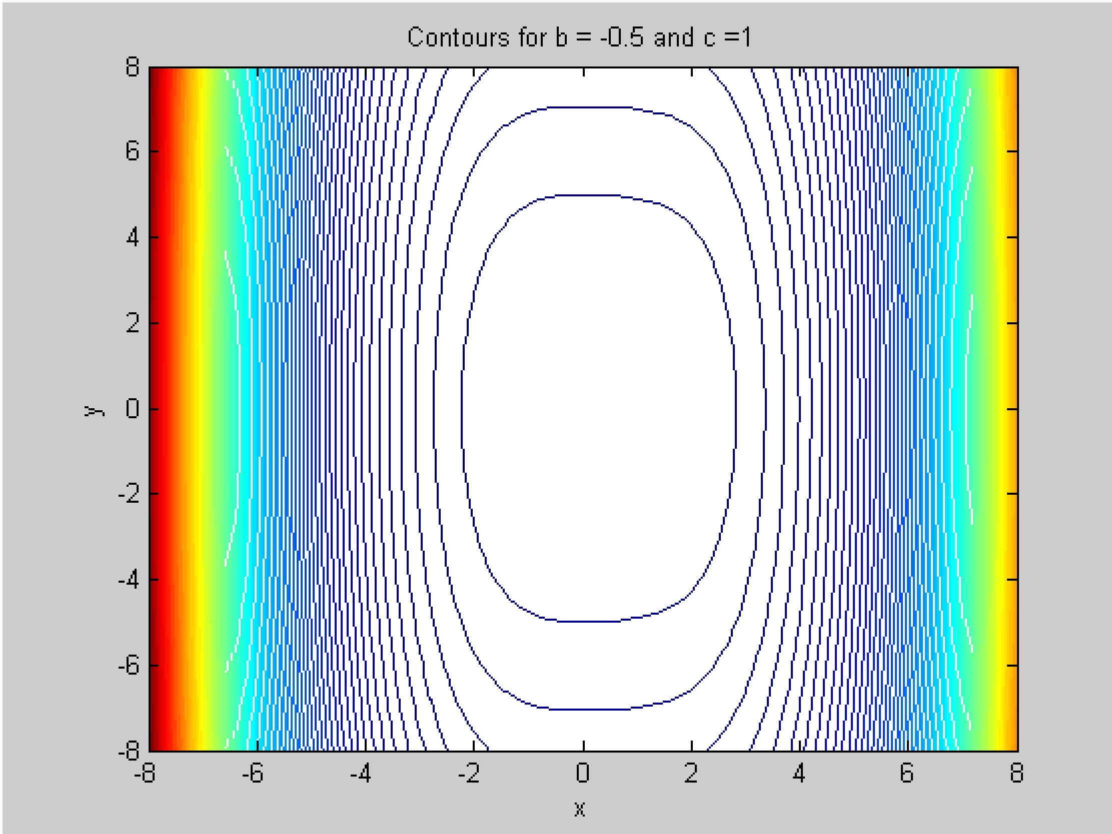

Contours for b = -0.5 and c =1
y
x
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2
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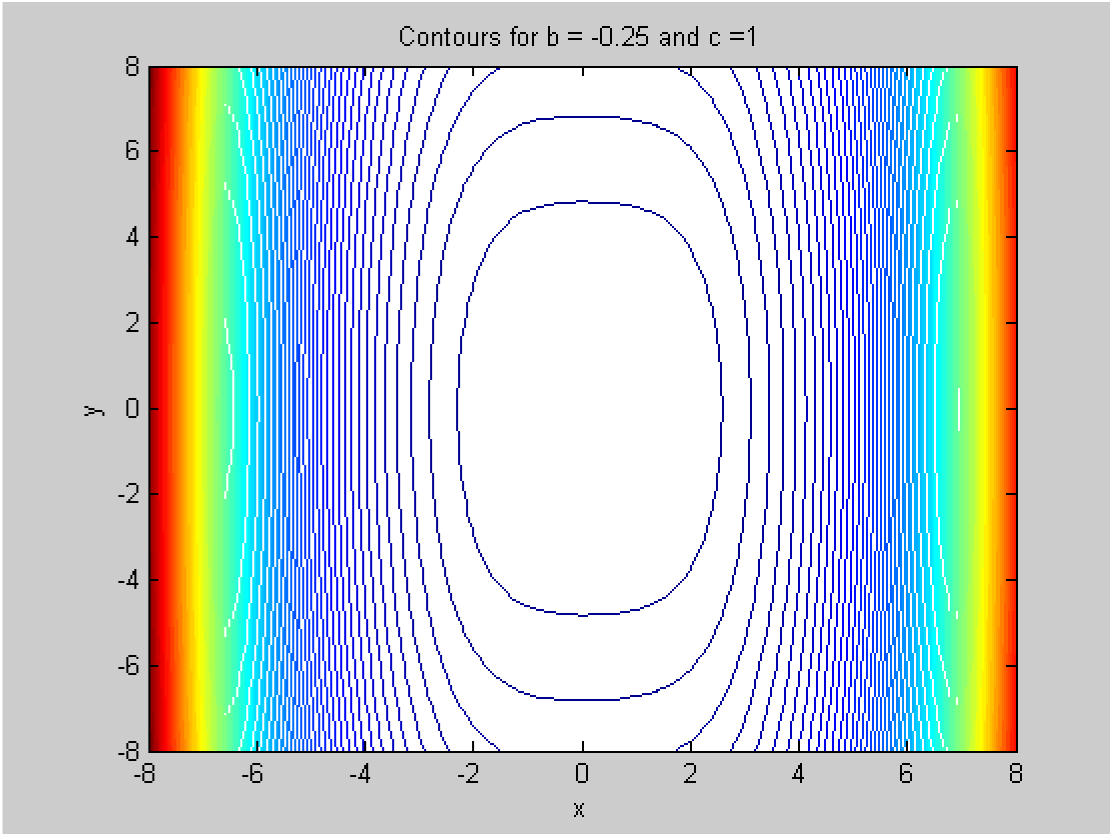
Contours for b = -0.25 and c =1
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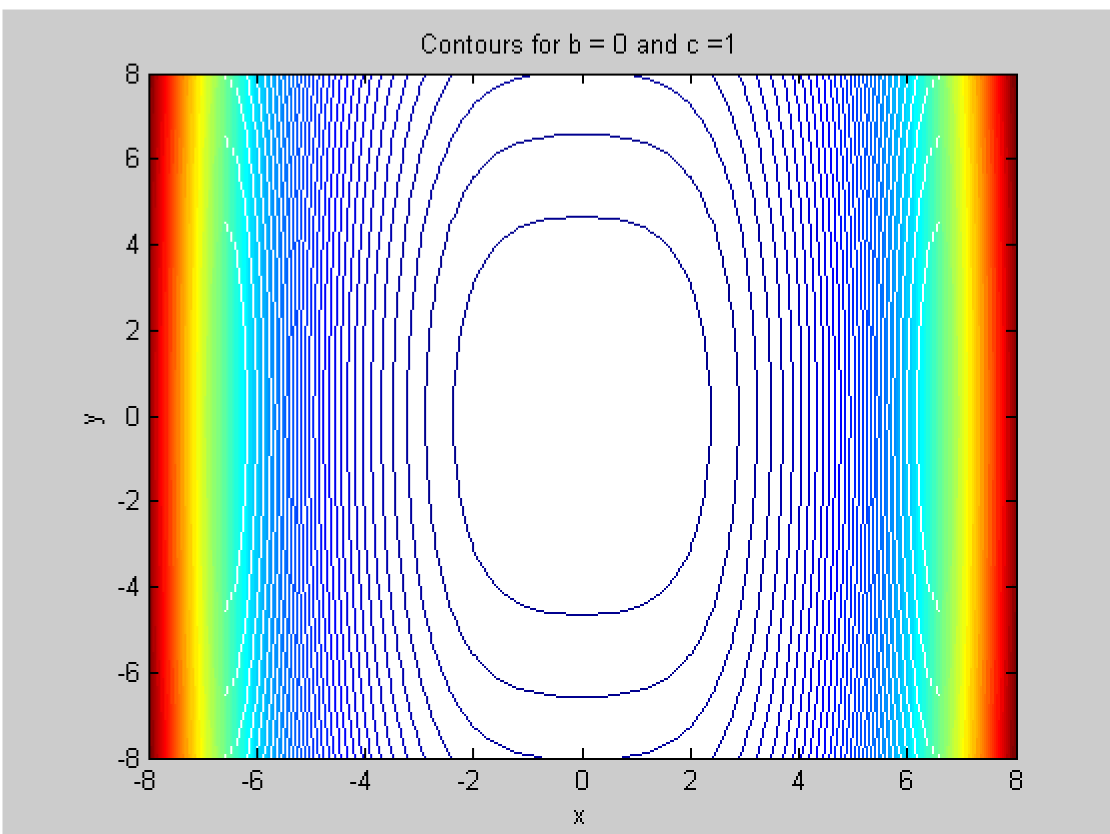
-6
-8
y
-8
-6
-4
-2
0
2
4
6
8
x


Contours for b = 0 and c =1
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y
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-2
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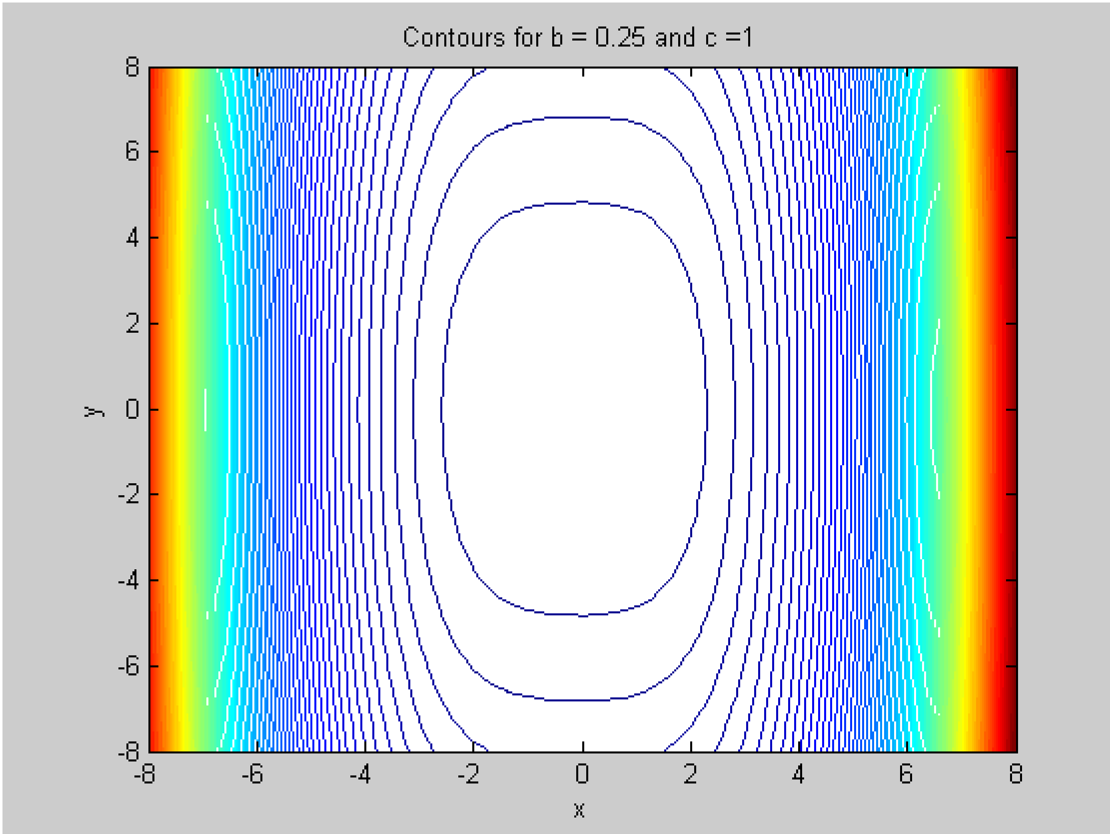
2
4
6
8
x



Contours for b = 0.25 and c =1
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y
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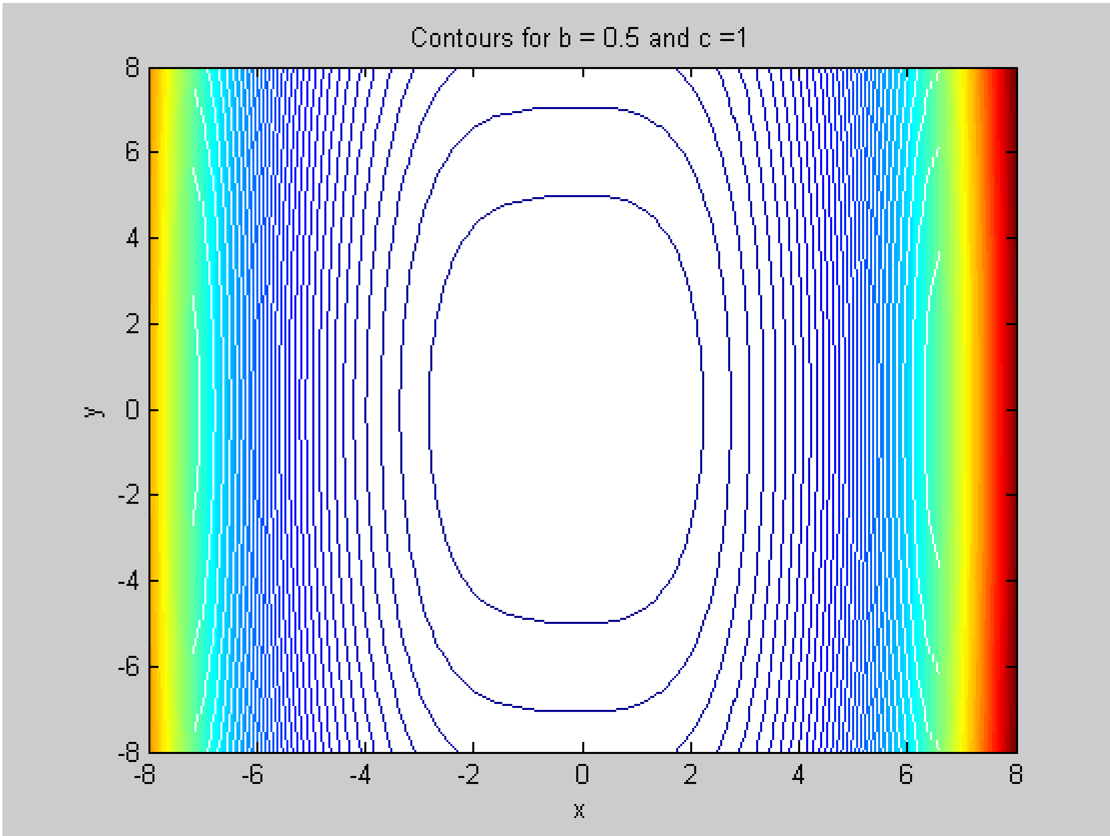
8
x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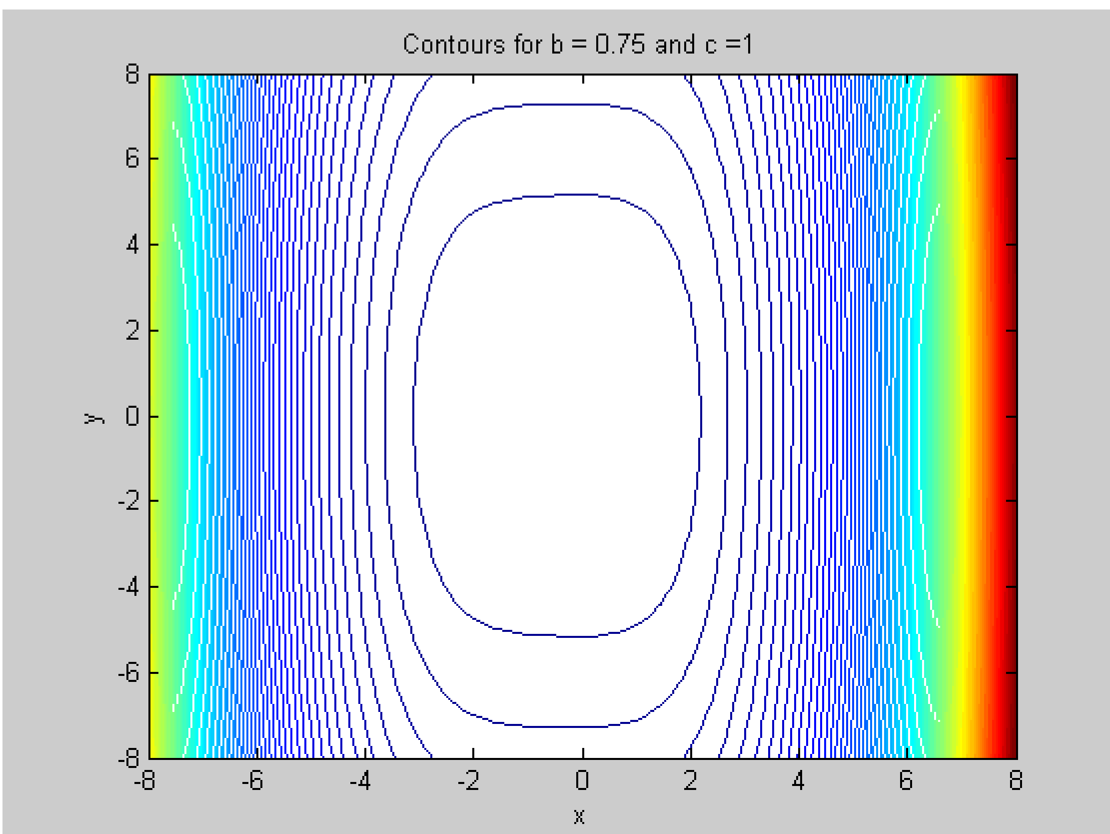
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x
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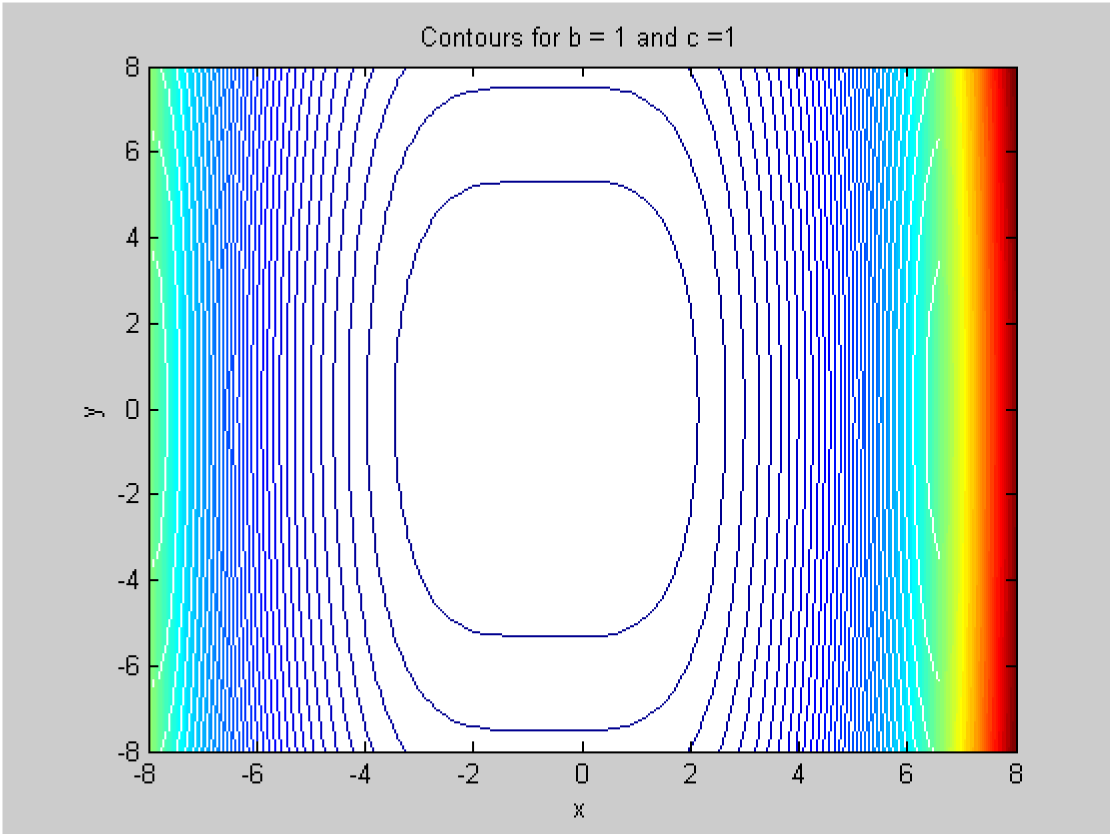
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y
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-2
0
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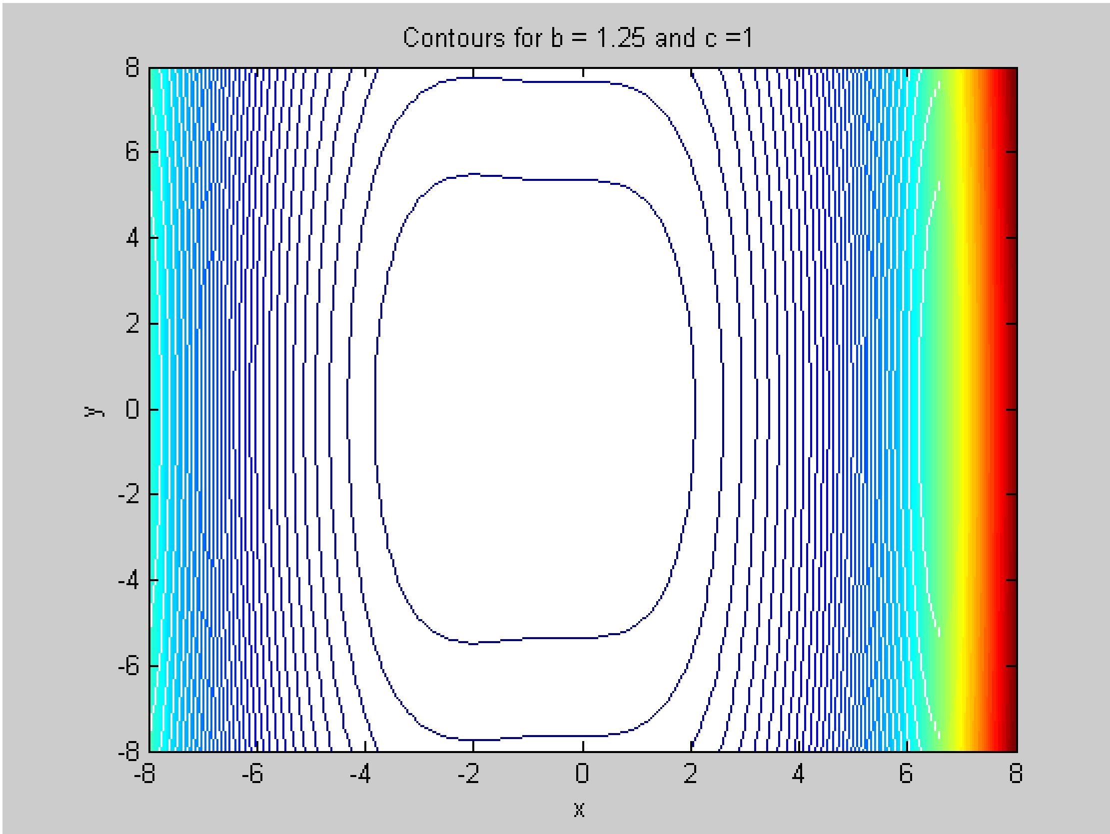
4
6
8
x


Contours for b = 1.25 and c =1
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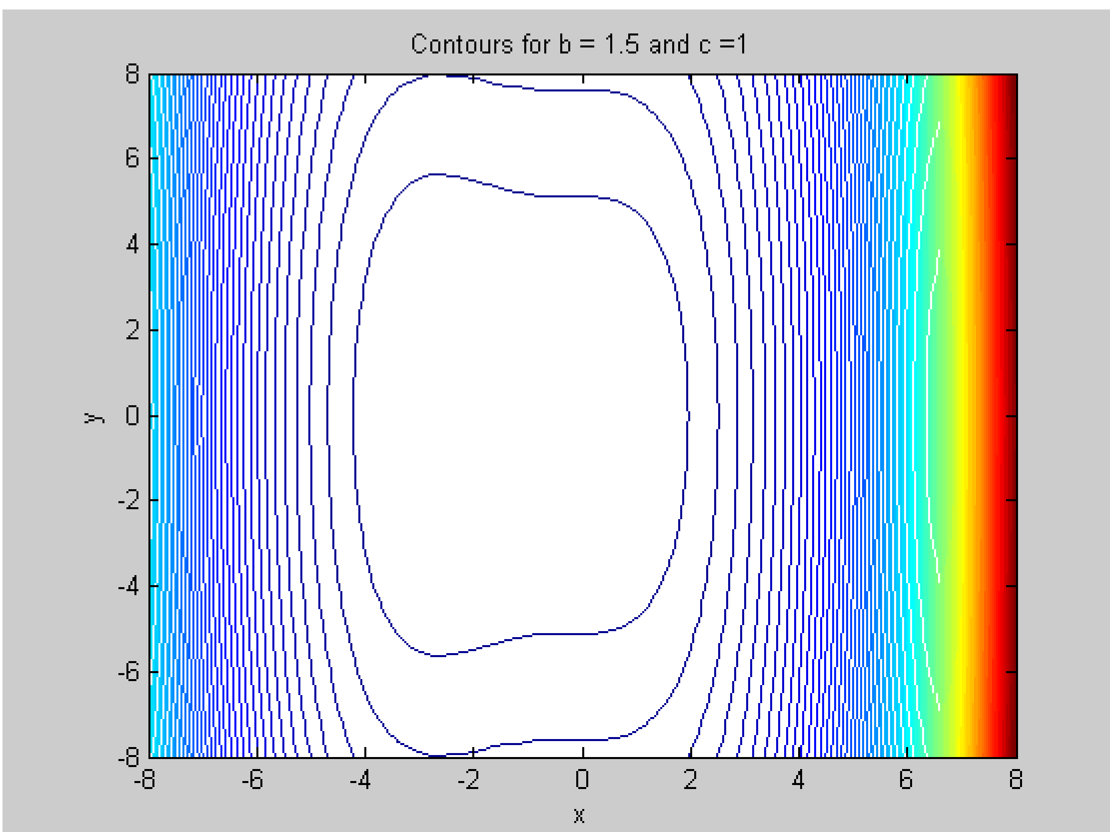
Contours for b = 1.5 and c =1
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x
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0
-2
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-4
-2
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2
4
6
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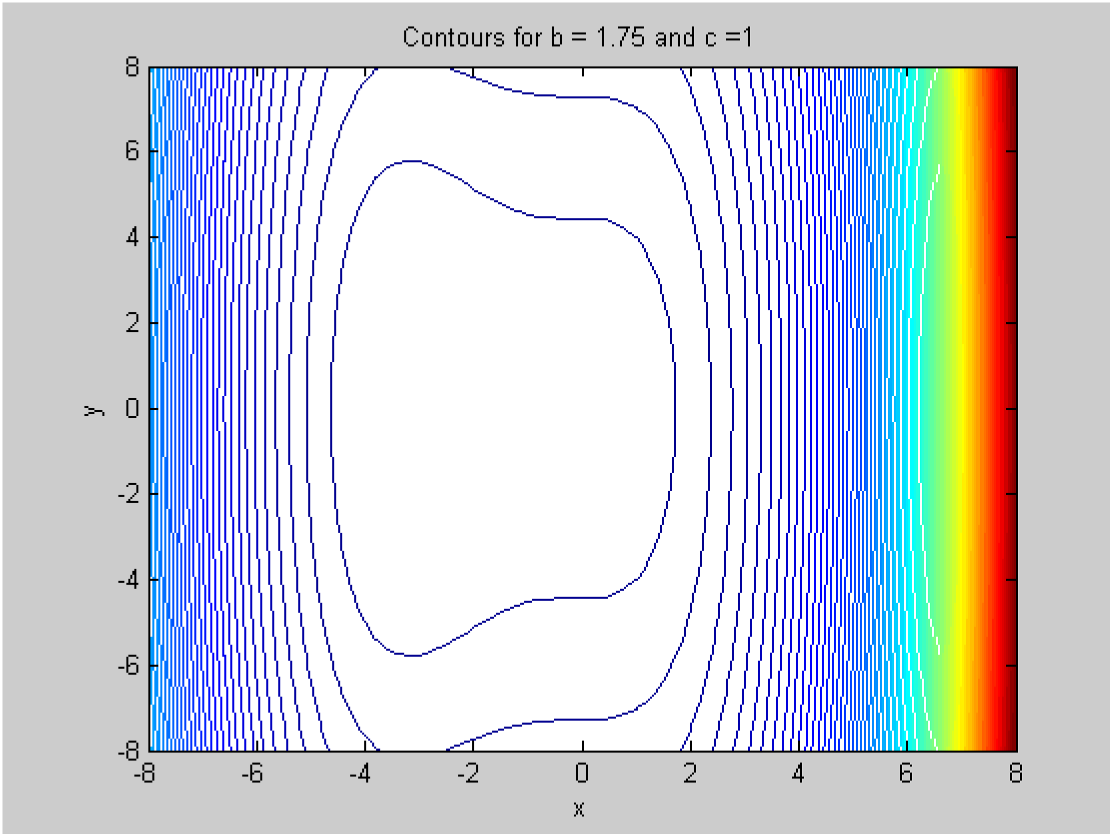
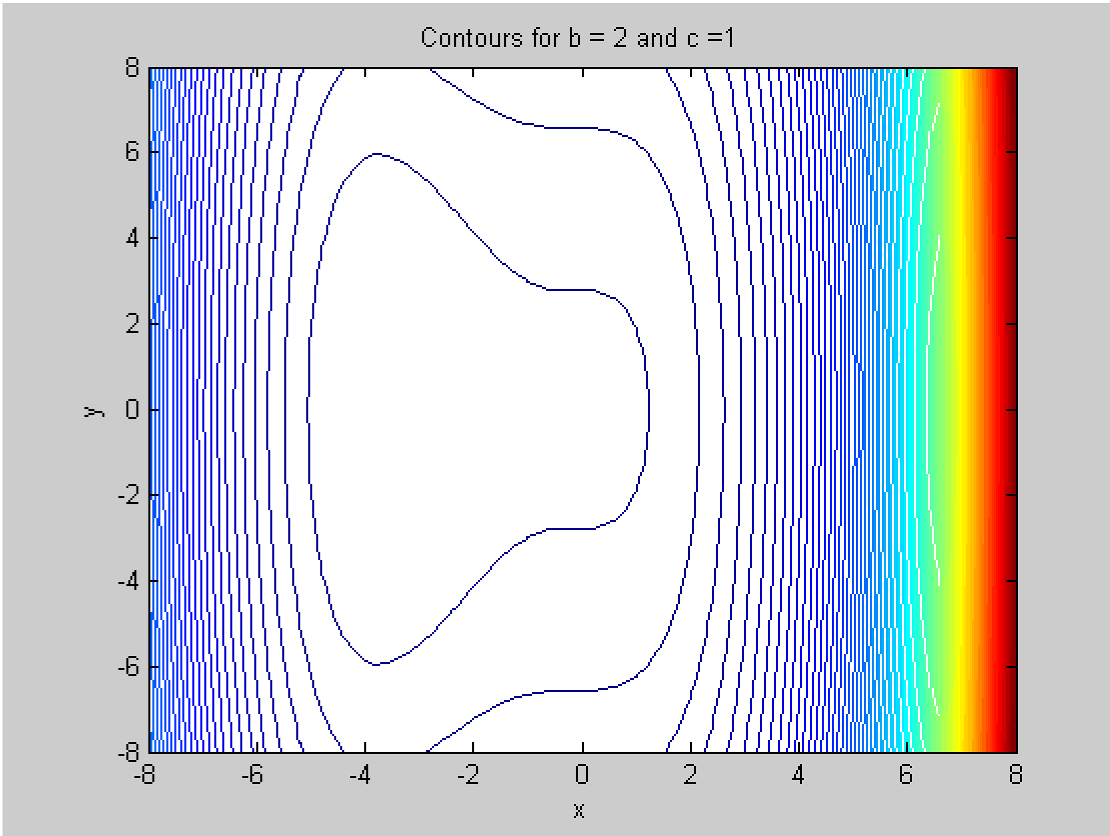


Contours for b = 1.75 and c =1
8
6
4
2
0
-2
-4
-6
-8
y
-8
-6
-4
-2
0
2
4
6
8
x


Contours for b = 2 and c =1
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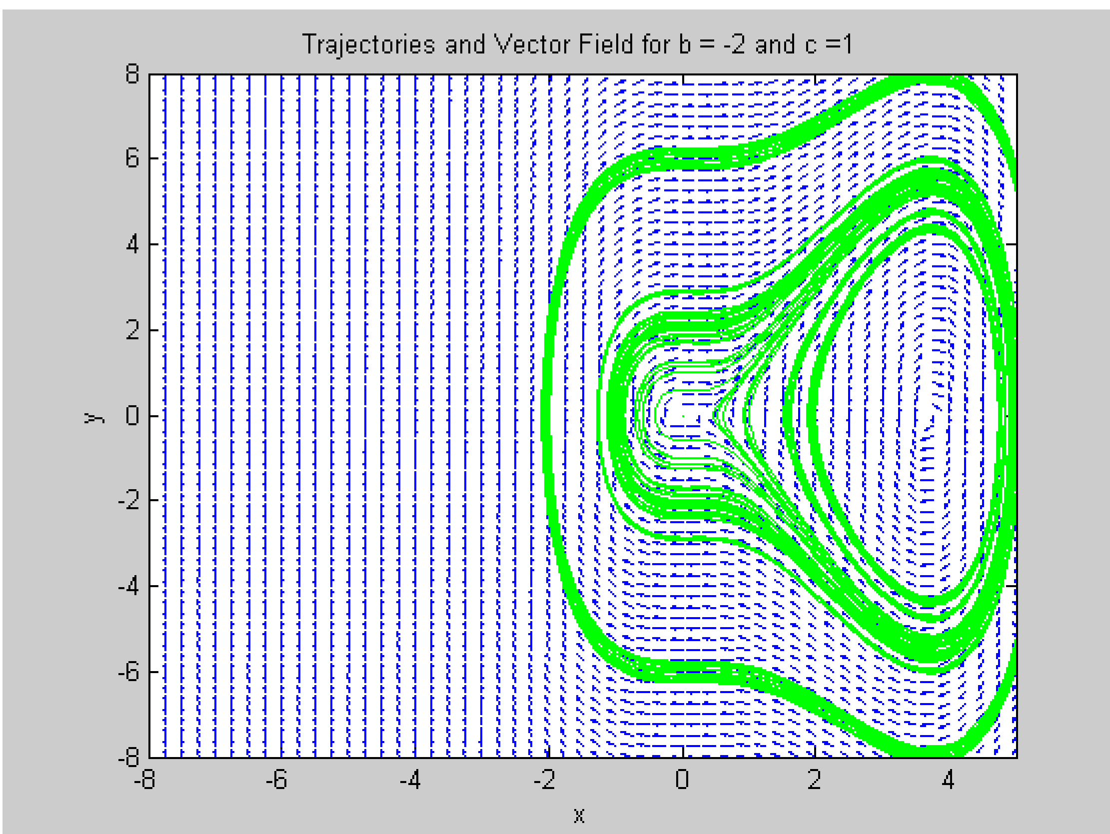
6
4
2
0
-2
-4
-6
-8
y
-8
-6
-4
-2
0
2
4
6
8
x


Trajectories and Vector Field for b = -2 and c =1
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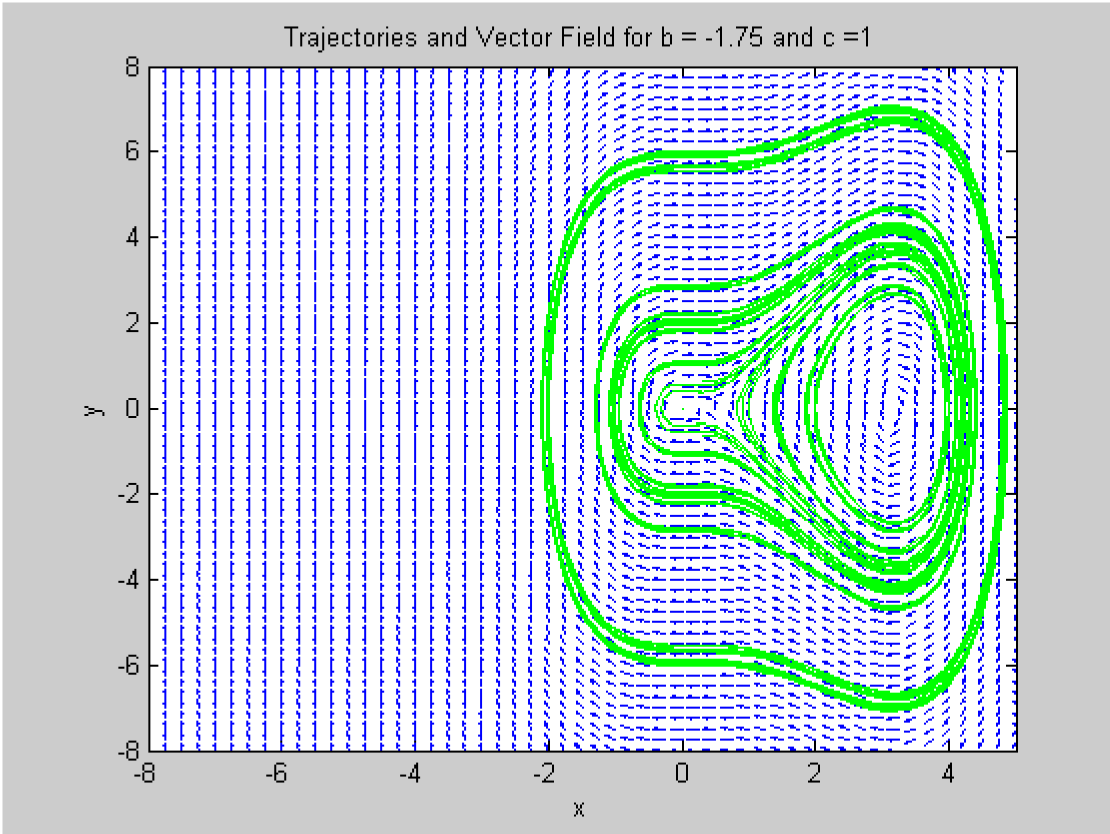
6
4
2
0
-2
-4
-6
-8
y
-8
-6
-4
-2
0
2
4
x



Trajectories and Vector Field for b = -1.75 and c =1
y
x
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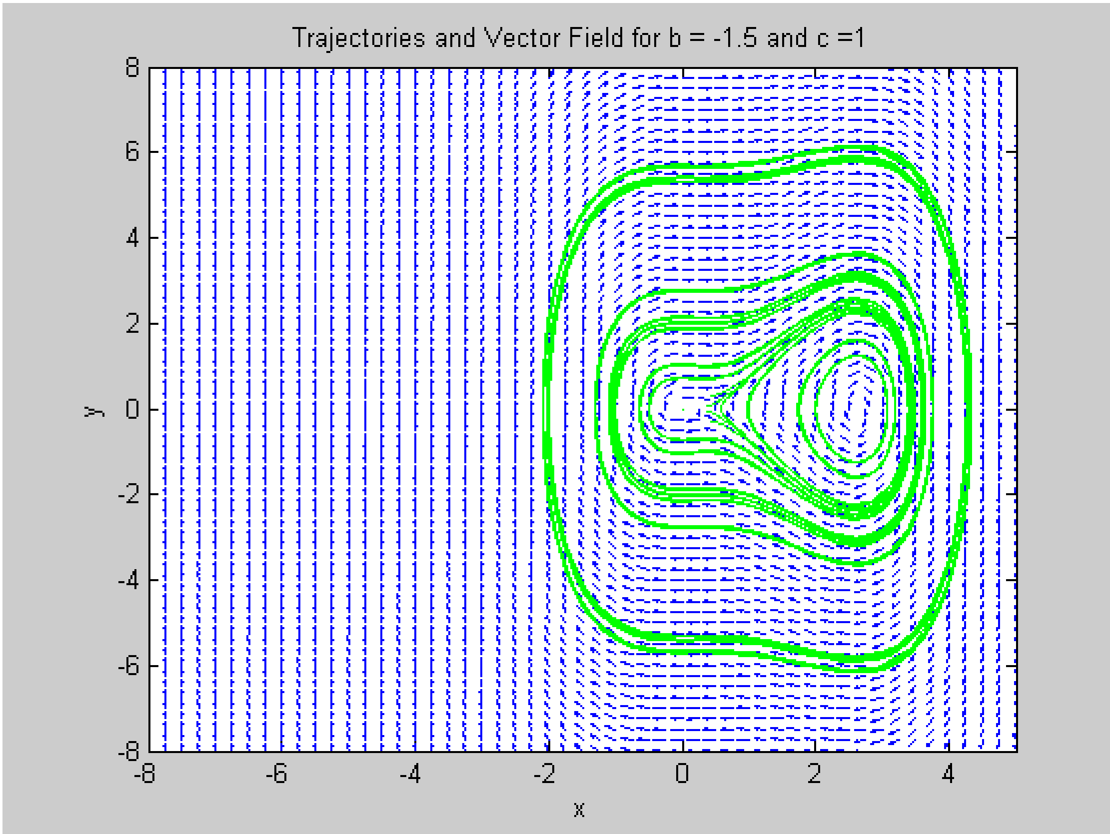
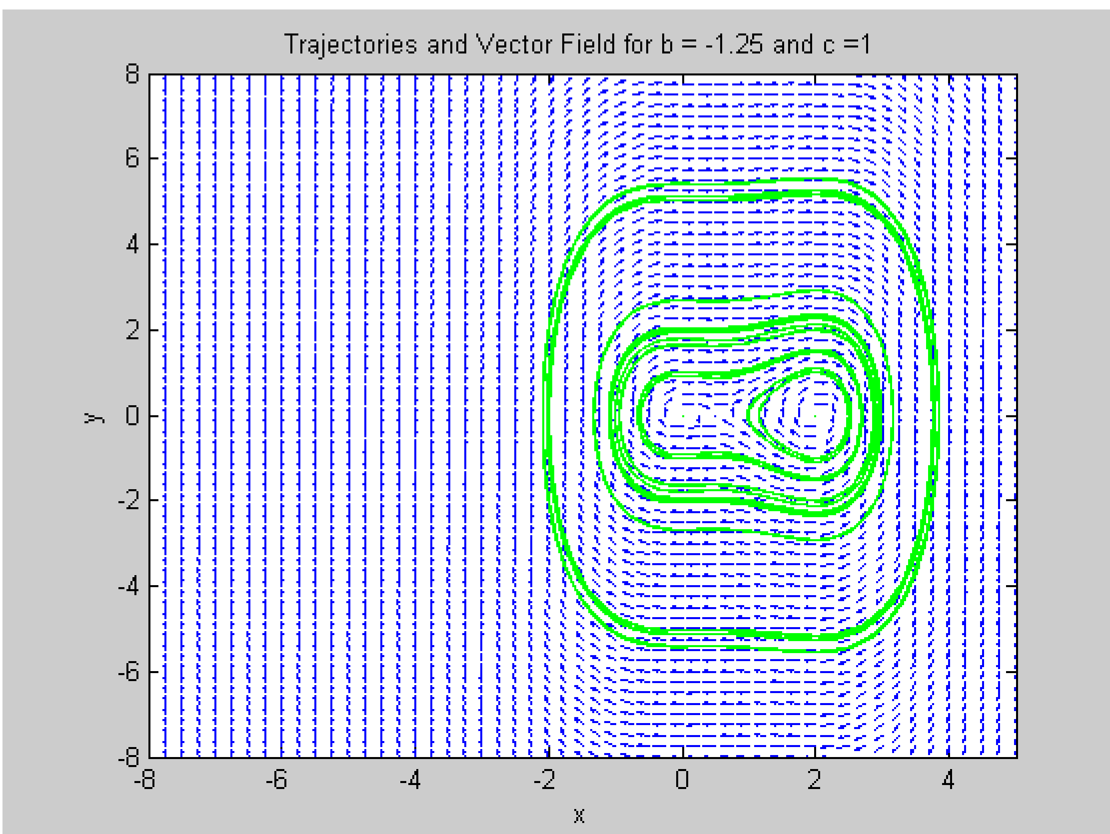
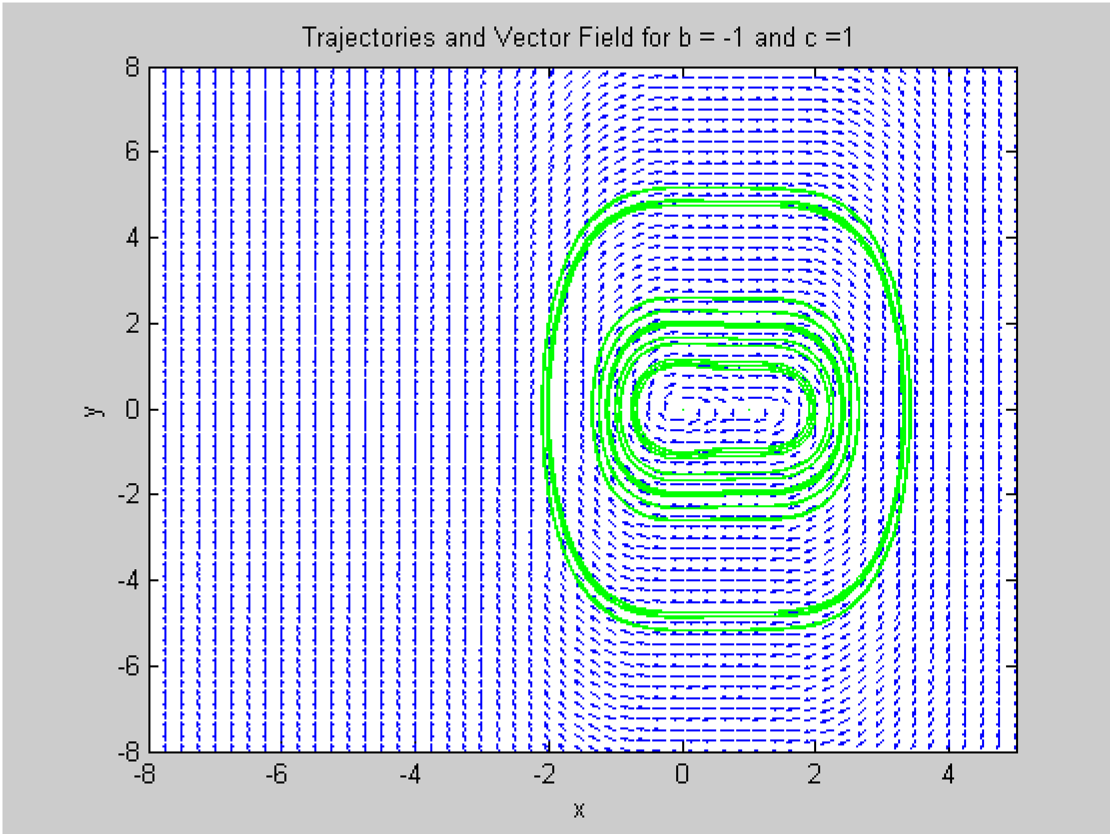
6
4
2
0
-2
-4
-6
-8
-8
-6
-4
-2
0
2
4

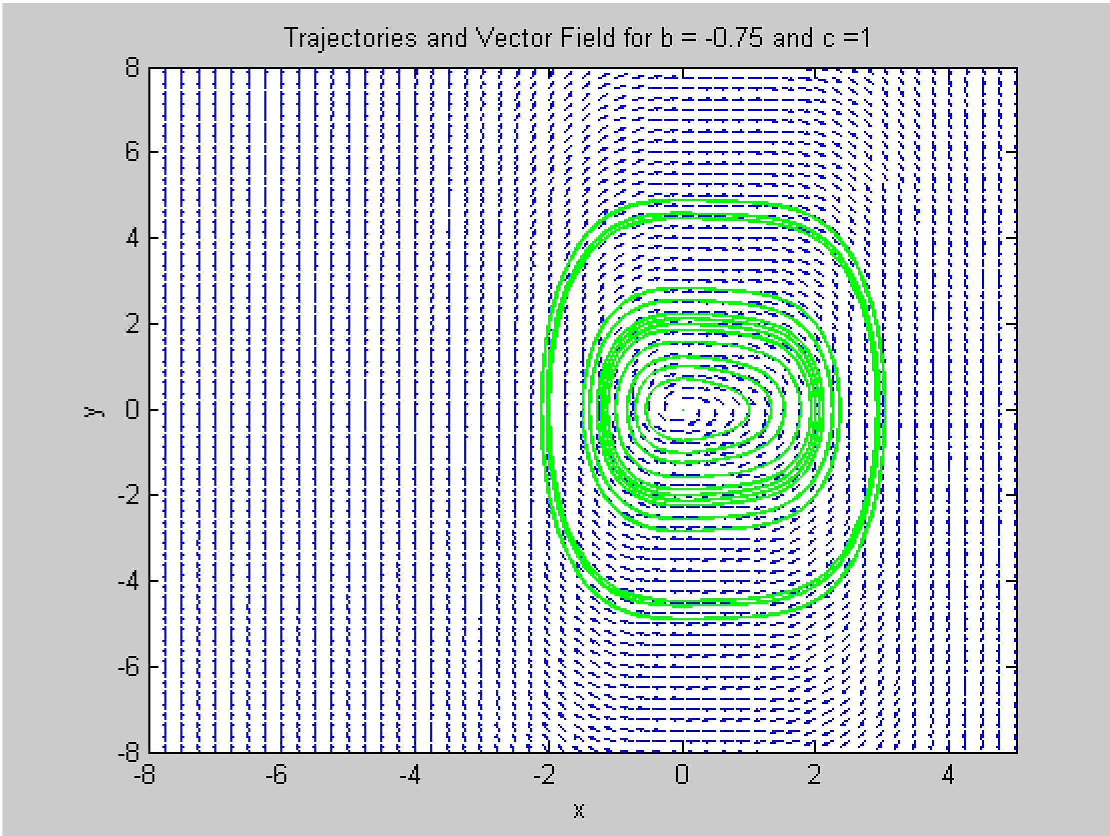


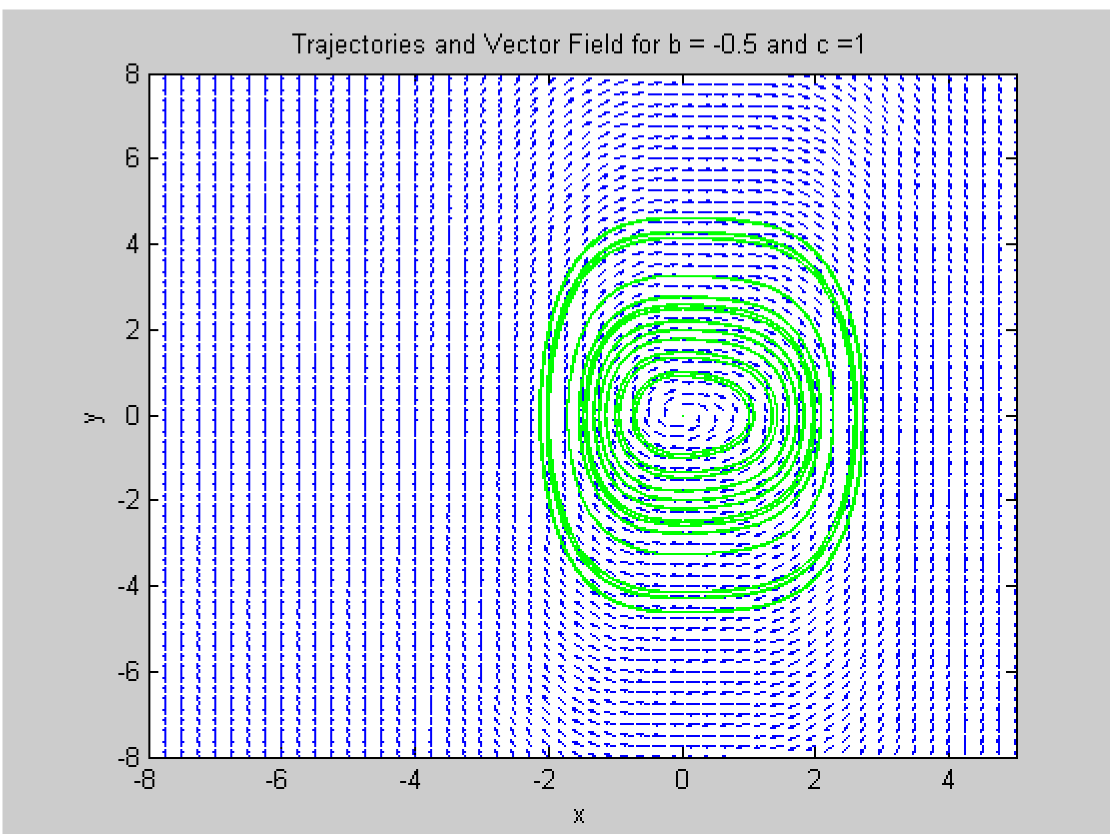
Trajectories and Vector Field for b = -1.5 and c =1
y
x

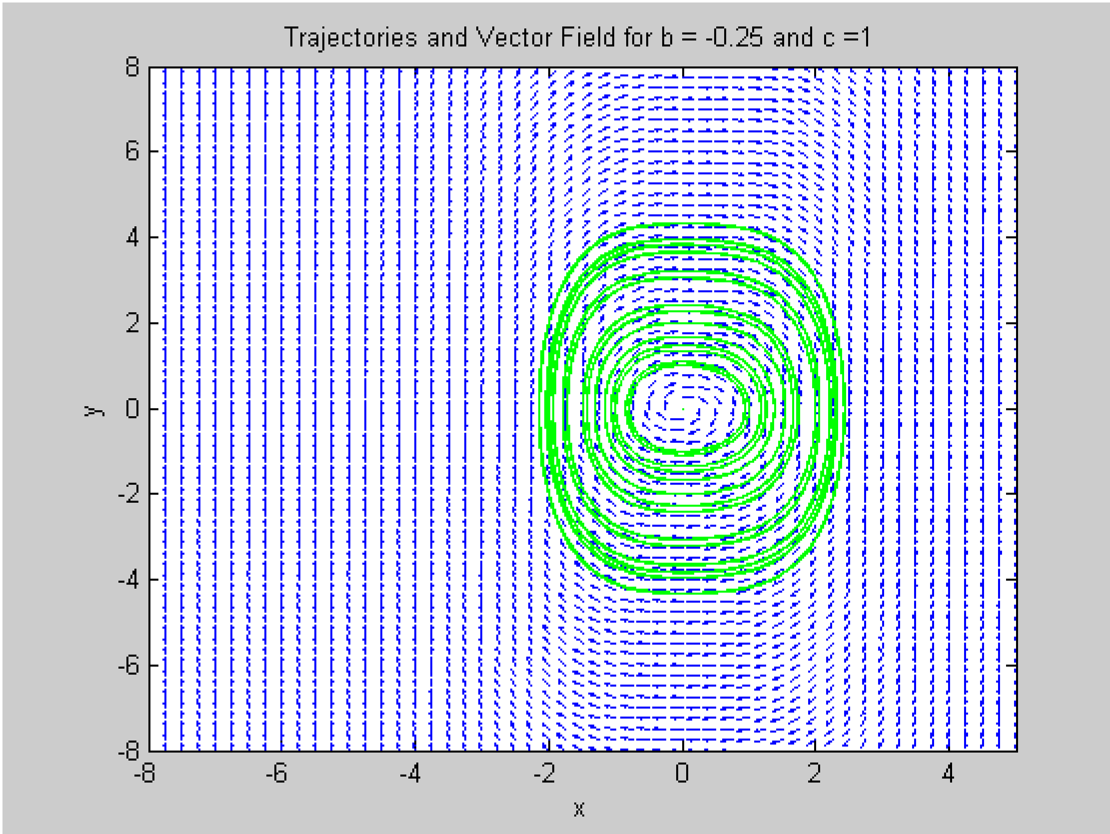


Trajectories and Vector Field for b = -1.25 and c =1
y
x

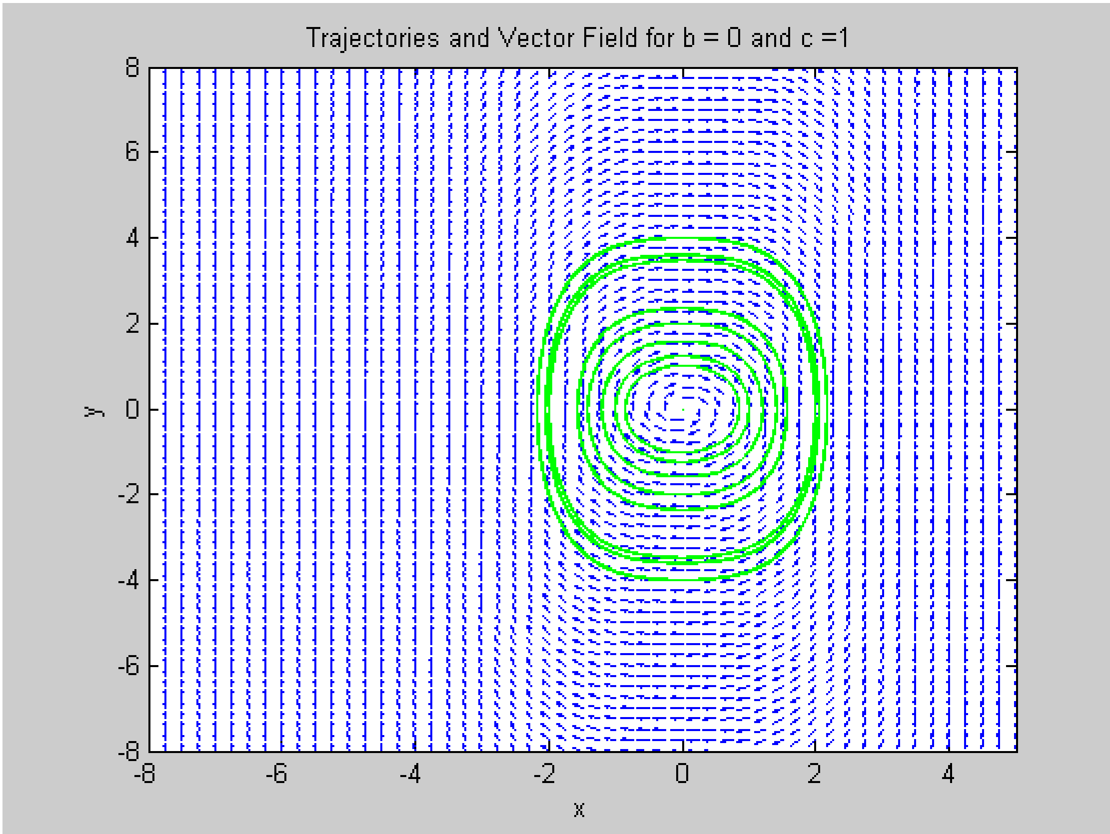

Trajectories and Vector Field for b = -1 and c =1
8
6
4
2
0
-2
-4
-6
-8
y
-8
-6
-4
-2
0
2
4
x

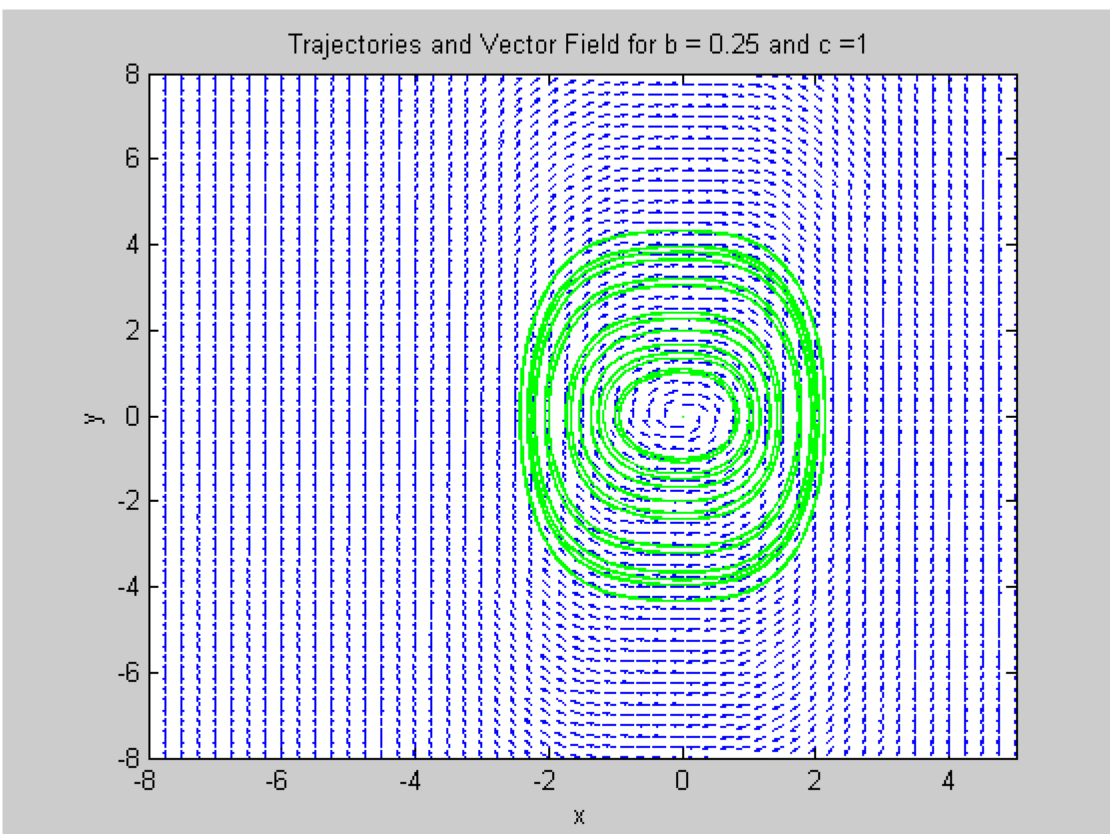

Trajectories and Vector Field for b = -0.75 and c =1
y
x
8
6
4
2
0
-2
-4
-6
-8
-8
-6
-4
-2
0
2
4

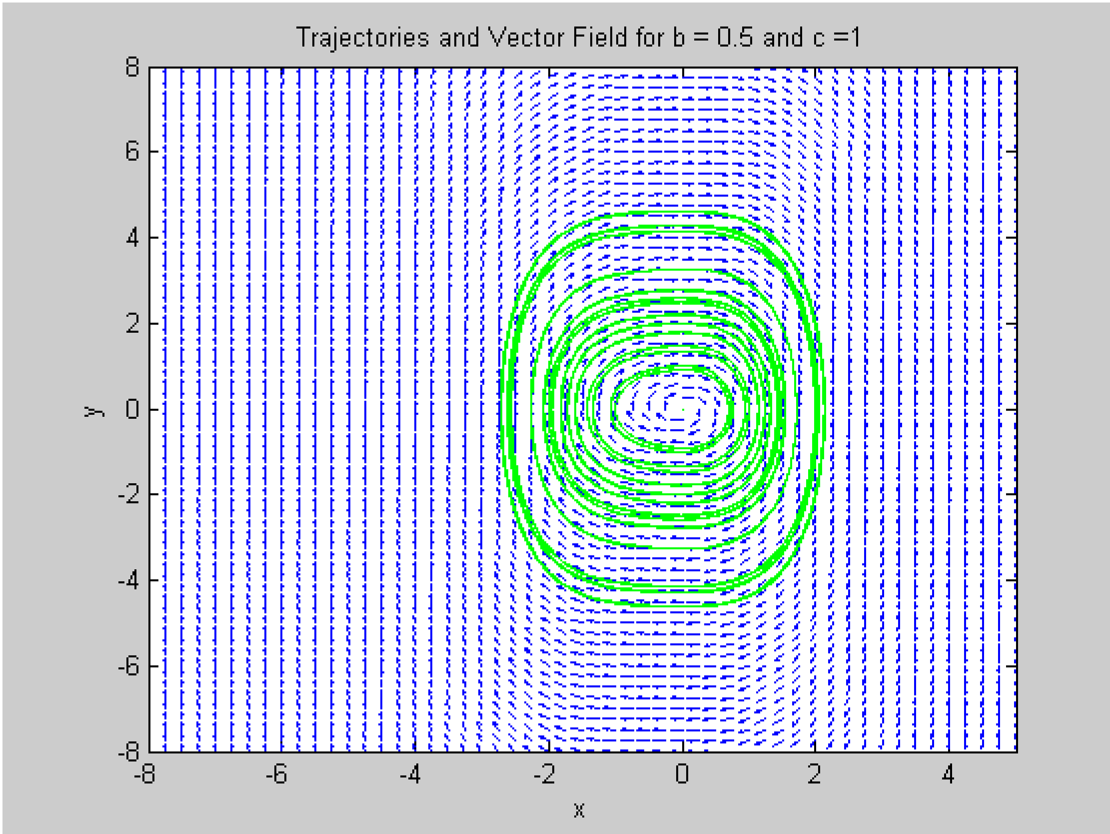

Trajectories and Vector Field for b = -0.5 and c =1
y
x
8
6
4
2
0
-2
-4
-6
-8
-8
-6
-4
-2
0
2
4

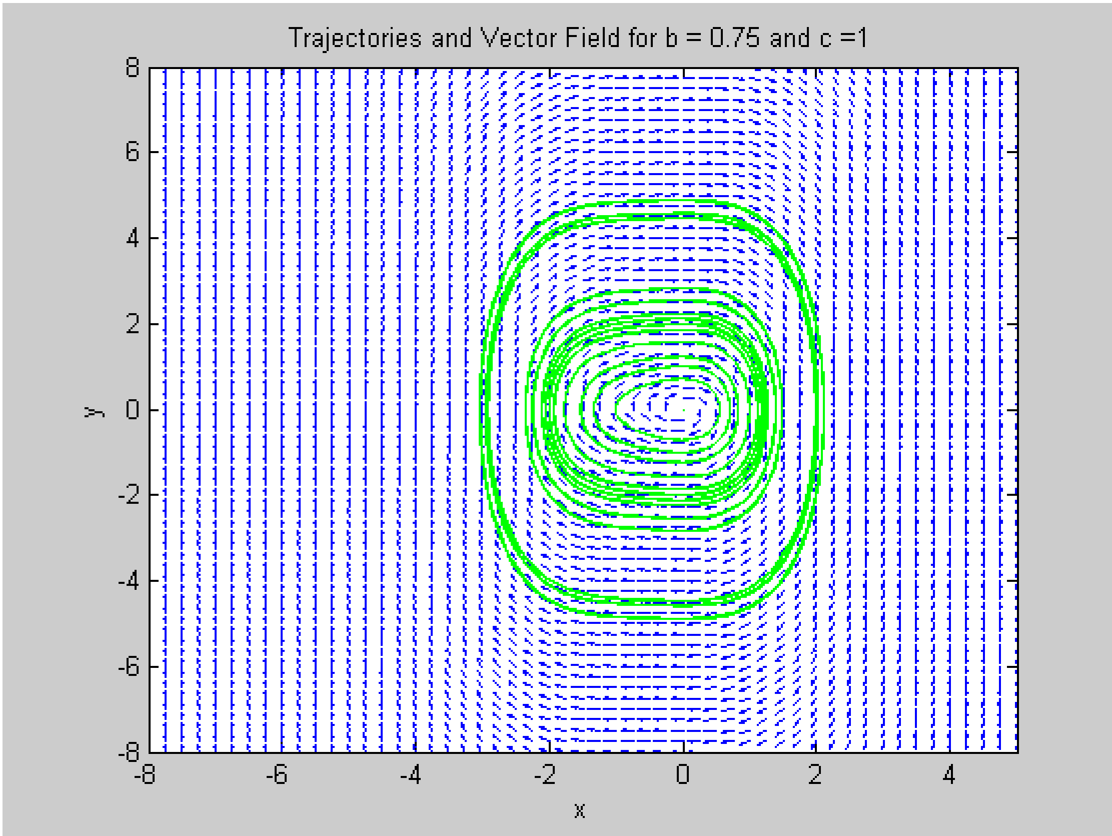


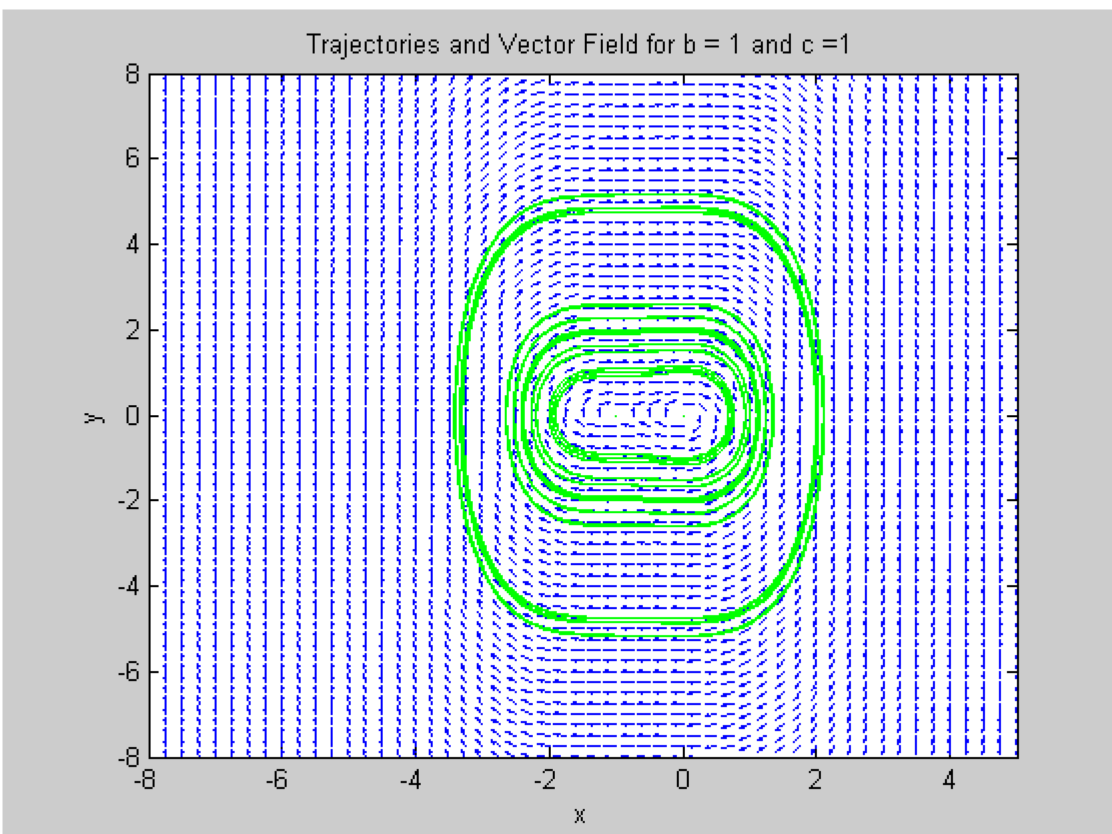
Trajectories and Vector Field for b = -0.25 and c =1
8
6
4
2
0
-2
-4
-6
-8
y
-8
-6
-4
-2
0
2
4
x

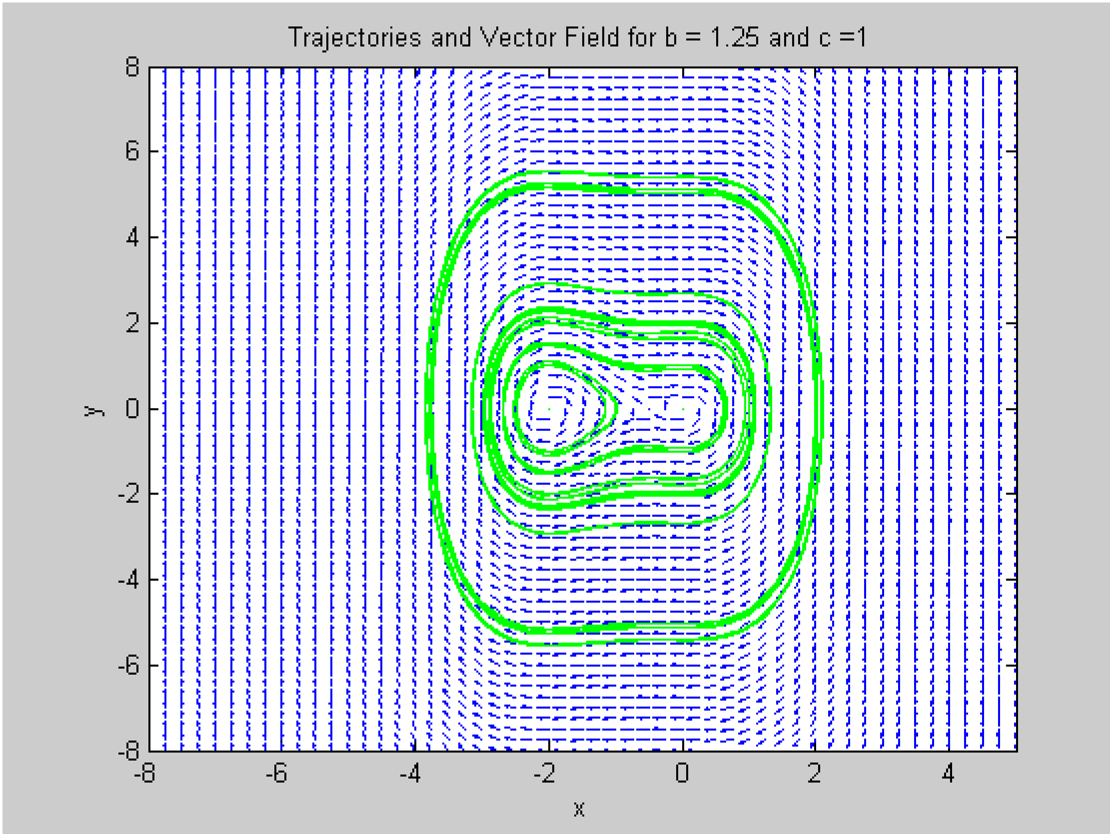

Trajectories and Vector Field for b = 0 and c =1
y
x

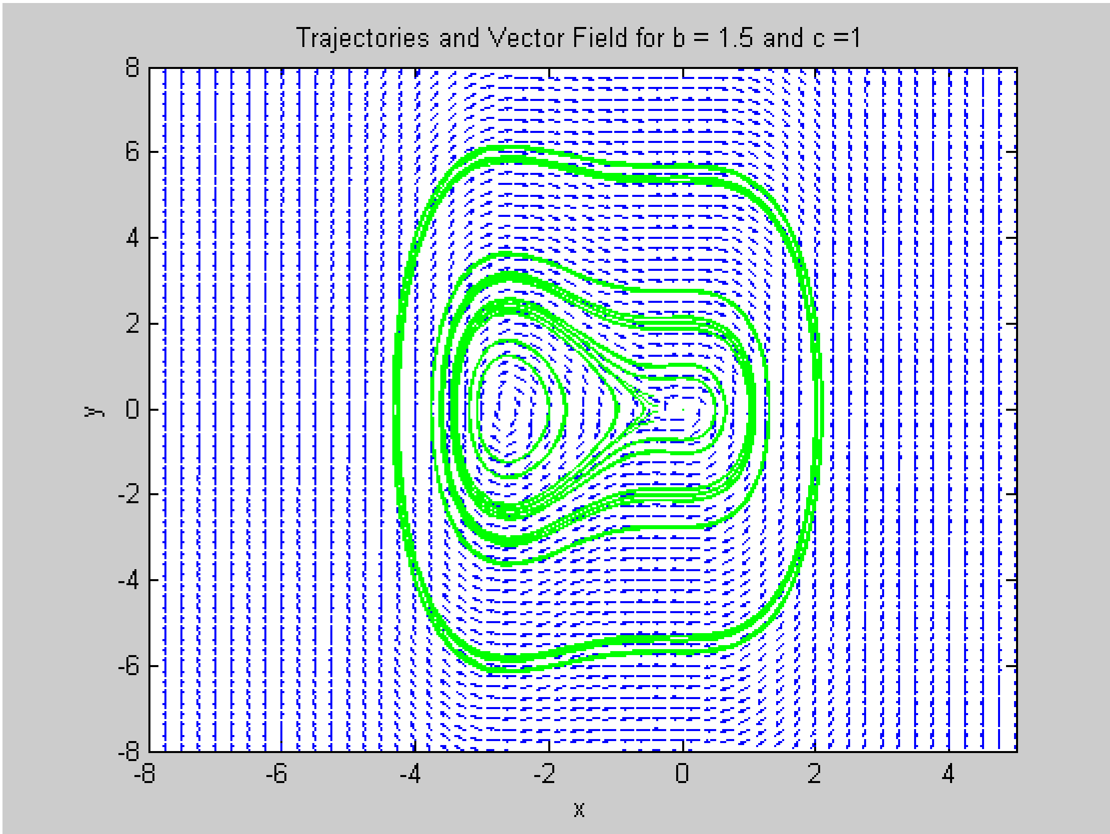

Trajectories and Vector Field for b = 0.25 and c =1
y
x

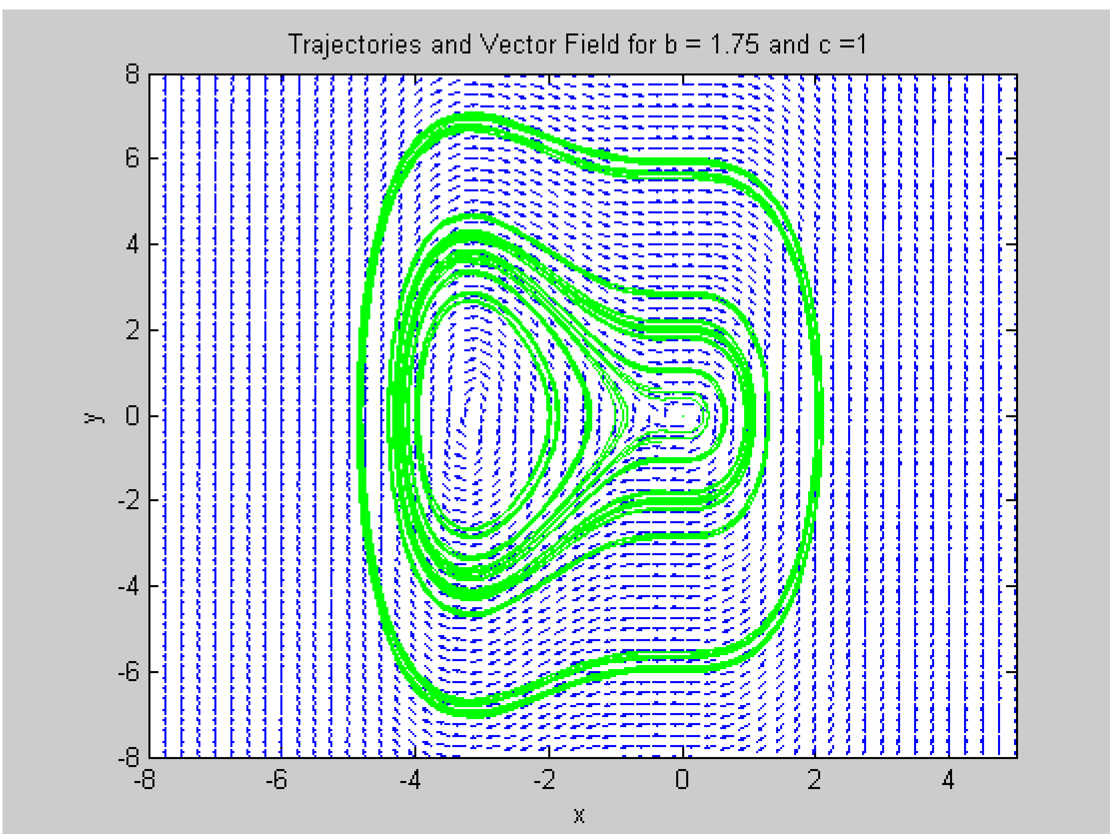

Trajectories and Vector Field for b = 0.5 and c =1
8
6
4
2
0
-2
-4
-6
-8
y
-8
-6
-4
-2
0
2
4
x

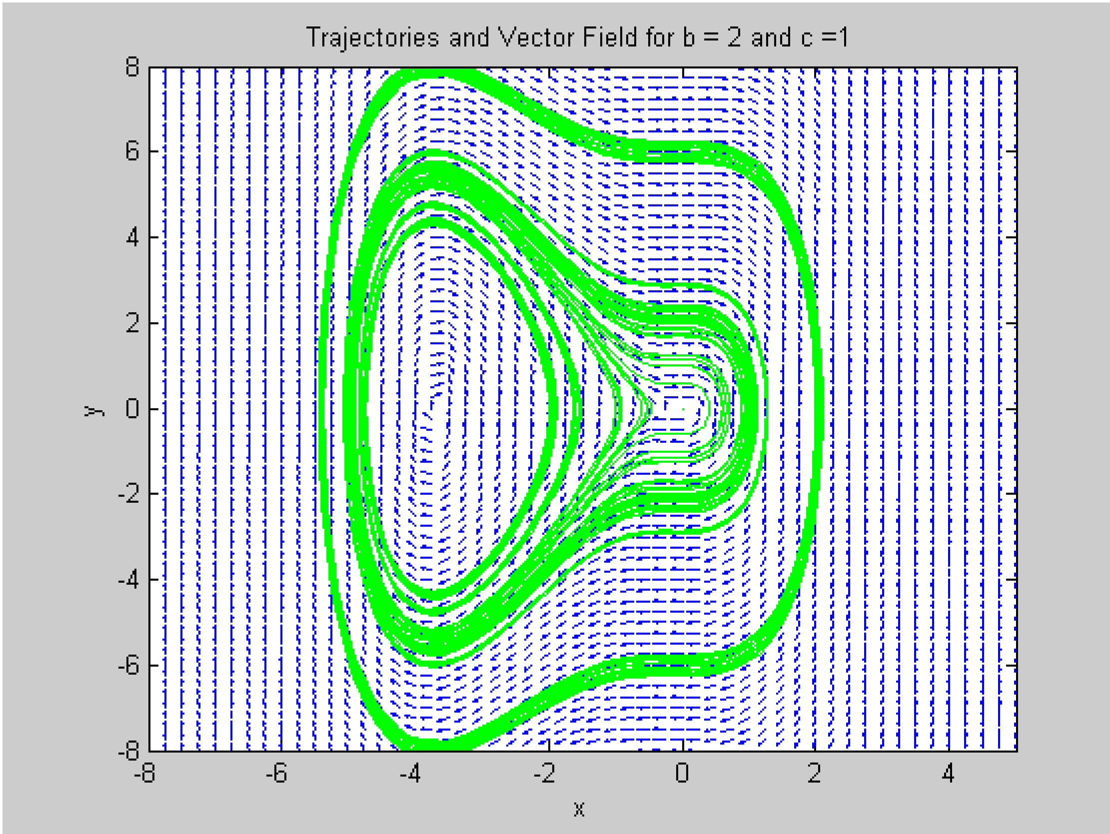

Trajectories and Vector Field for b = 0.75 and c =1
y
x


Trajectories and Vector Field for b = 1 and c =1
y
x


Trajectories and Vector Field for b = 1.25 and c =1
8
6
4
2
0
-2
-4
-6
-8
y
-8
-6
-4
-2
0
2
4
x


Trajectories and Vector Field for b = 1.5 and c =1
y
x
8
6
4
2
0
-2
-4
-6
-8
-8
-6
-4
-2
0
2
4


Trajectories and Vector Field for b = 1.75 and c =1
y
x
8
6
4
2
0
-2
-4
-6
-8
-8
-6
-4
-2
0
2
4



Trajectories and Vector Field for b = 2 and c =1
8
6
4
2
0
-2
-4
-6
-8
y
-8
-6
-4
-2
0
2
4
x

Summary of Critical Points and Stability for Various b Values (continued on next page)

b	-2			
Critical Points		(0,0)	(3.73,0)	(0.27,0)
Eigenvalues		i -i	3.60i -3.60i	0.96 -0.96
Description		purely imaginary pair	purely imaginary pair	real, one + one -
Type of Critical Point		Stable Center	Stable Center	Unstable Saddle Point
b	**-1.75**			
Critical Points		(0,0)	(3.19,0)	(0.31,0)
Eigenvalues		i -i	3.03i -3.03i	0.95 -0.95
Description		purely imaginary pair	purely imaginary pair	real, one + one -
Type of Critical Point		Stable Center	Stable Center	Unstable Saddle Point
b	**-1.5**			
Critical Points		(0,0)	(2.62,0)	(0.38,0)
Eigenvalues		i -i	2.42i -2.42i	0.92 -0.92
Description		purely imaginary pair	purely imaginary pair	real, one + one -
Type of Critical Point		Stable Center	Stable Center	Unstable Saddle Point
b	**-1.25**			
Critical Points		(0,0)	(2.00,0)	(0.50,0)
Eigenvalues		i -i	1.73i -1.73i	0.87 -0.87
Description		purely imaginary pair	purely imaginary pair	real, one + one -
Type of Critical Point		Stable Center	Stable Center	Unstable Saddle Point
b	**-1**			
Critical Points		(0,0)	(1,0)	(1,0)
Eigenvalues		i -i	0 0	0 0
Description		purely imaginary pair	zeros, identical	zeros, identical
Type of Critical Point		Stable Center	Stable Zero	Stable Zero
b	**-0.75**			
Critical Points		(0,0)	(0.75 + 0.66i,0)	(0.75 - 0.66i,0)
Eigenvalues		i -i	.94 -.94	.94 -.94
Description		purely imaginary pair	real, one + one -	real, one + one -
Type of Critical Point		Stable Center	Unstable Saddle Point	Unstable Saddle Point
b	**-0.5**			
Critical Points		0	(0.5 + 0.87i,0)	(0.5 - 0.87i,0)
Eigenvalues		(0,0) -i	1.22 -1.22	1.22 -1.22
Description		purely imaginary pair	real, one + one -	real, one + one -
Type of Critical Point		Stable Center	Unstable Saddle Point	Unstable Saddle Point
b	**-0.25**			
Critical Points		(0,0)	(0.25 + 0.97i,0)	(0.25 - 0.97i,0)
Eigenvalues		i -i	1.37 -1.37	1.37 -1.37
Description		purely imaginary pair	real, one + one -	real, one + one -
Type of Critical Point		Stable Center	Unstable Saddle Point	Unstable Saddle Point

b	**0**		
Critical Points	(0,0)	(i,0)	(-i,0)
Eigenvalues	i	1.41	1.41
	-i	-1.41	-1.41
Description	purely imaginary pair	real, one + one -	real, one + one -
Type of Critical Point	Stable Center	Unstable Saddle Point	Unstable Saddle Point
b	**0.25**		
Critical Points	(0,0)	(-0.25 + 0.97i,0)	(-0.25 - 0.97i,0)
Eigenvalues	i	1.37	1.37
	-i	-1.37	-1.37
Description	purely imaginary pair	real, one + one -	real, one + one -
Type of Critical Point	Stable Center	Unstable Saddle Point	Unstable Saddle Point
b	**0.5**		
Critical Points	(0,0)	(-0.5 + 0.87i,0)	(-0.5 - 0.87i,0)
Eigenvalues	i	1.22	1.22
	-i	-1.22	-1.22
Description	purely imaginary pair	real, one + one -	real, one + one -
Type of Critical Point	Stable Center	Unstable Saddle Point	Unstable Saddle Point
b	**0.75**		
Critical Points	(0,0)	(-0.75 + 0.66i,0)	(-0.75 - 0.66i,0)
Eigenvalues	i	0.94	0.94
	-i	-0.94	-0.94
Description	purely imaginary pair	real, one + one -	real, one + one -
Type of Critical Point	Stable Center	Unstable Saddle Point	Unstable Saddle Point
b	**1**		
Critical Points	(0,0)	(-1,0)	(-1,0)
Eigenvalues	i	0	0
	-i	0	0
Description	purely imaginary pair	zeros, identical	zeros, identical
Type of Critical Point	Stable Center	Stable Zero	Stable Zero
b	**1.25**		
Critical Points	(0,0)	(-.50,0)	(-2.00,0)
Eigenvalues	i	0.87	1.73i
	-i	-0.87	-1.73i
Description	purely imaginary pair	real, one + one -	purely imaginary pair
Type of Critical Point	Stable Center	Unstable Saddle Point	Stable Center
b	**1.5**		
Critical Points	(0,0)	(-0.38,0)	(-2.62,0)
Eigenvalues	i	0.92	2.42i
	-i	-0.92	-2.42i
Description	purely imaginary pair	real, one + one -	purely imaginary pair
Type of Critical Point	Stable Center	Unstable Saddle Point	Stable Center
b	**1.75**		
Critical Points	(0,0)	(-0.31,0)	(-3.19,0)
Eigenvalues	i	0.95	3.03i
	-i	-0.95	-3.03i
Description	purely imaginary pair	real, one + one -	purely imaginary pair
Type of Critical Point	Stable Center	Unstable Saddle Point	Stable Center
b	**2**		
Critical Points	(0,0)	(-0.27,0)	(-3.73,0)
Eigenvalues	i	0.96	3.60i
	-i	-0.96	-3.60i
Description	purely imaginary pair	real, one + one -	purely imaginary pair
Type of Critical Point	Stable Center	Unstable Saddle Point	Stable Center