Solutions to Homework Problems on the Green Function Method II
Spring 2009, Math 246, Professor David Levermore

1. Compute the Green function $G(t,s)$ for the differential operator $\mathrm{L}(t)$ defined by
$$\mathrm{L}(t)y = \mathrm{D}^2 y - 2t\mathrm{D}y + (t^2-1)y\,,$$
given that $e^{\frac{1}{2}t^2}$ and $te^{\frac{1}{2}t^2}$ solve the homogeneous equation $\mathrm{L}(t)y = 0$. Use the result to solve the initial-value problem
$$y'' - 2ty' + (t^2-1)y = t^2 e^{\frac{1}{2}t^2}\,, \quad y(0) = y'(0) = 0\,.$$

Solution: The Green function is given by
$$G(t,s) = \frac{\det\begin{pmatrix} e^{\frac{1}{2}s^2} & se^{\frac{1}{2}s^2} \\ e^{\frac{1}{2}t^2} & te^{\frac{1}{2}t^2} \end{pmatrix}}{\det\begin{pmatrix} e^{\frac{1}{2}s^2} & se^{\frac{1}{2}s^2} \\ se^{\frac{1}{2}s^2} & (s^2+1)e^{\frac{1}{2}s^2} \end{pmatrix}} = (t-s)e^{\frac{1}{2}t^2}e^{-\frac{1}{2}s^2}\,.$$
Because the equation is in normal form the forcing is $t^2 e^{\frac{1}{2}t^2}$. The solution is therefore
$$y(t) = \int_0^t G(t,s)\, s^2 e^{\frac{1}{2}s^2}\, \mathrm{d}s = e^{\frac{1}{2}t^2}\int_0^t (t-s)s^2\,\mathrm{d}s = \tfrac{1}{12}t^4 e^{\frac{1}{2}t^2}\,.$$

2. Compute the Green function $G(t,s)$ for the differential operator $\mathrm{L}(t)$ defined by
$$\mathrm{L}(t)y = t\,\mathrm{D}^2 y + (t-1)\mathrm{D}y - y\,,$$
given that $t-1$ and e^{-t} solve the homogeneous equation $\mathrm{L}(t)y = 0$. Use the result to solve the initial-value problem
$$t\,y'' + (t-1)y' - y = 2t^3\,, \quad y(1) = y'(1) = 0\,.$$

Solution: The Green function is given by
$$G(t,s) = \frac{\det\begin{pmatrix} s-1 & e^{-s} \\ t-1 & e^{-t} \end{pmatrix}}{\det\begin{pmatrix} s-1 & e^{-s} \\ 1 & -e^{-s} \end{pmatrix}} = \frac{(t-1)e^{-s} - e^{-t}(s-1)}{s\,e^{-s}}\,.$$
Because the normal form of the equation is
$$y'' + \left(1 - \frac{1}{t}\right)y' - \frac{1}{t}\,y = 2t^2\,,$$
the forcing is $2t^2$. The solution is therefore
$$y(t) = \int_1^t G(t,s)\,2s^2\,\mathrm{d}s = 2\int_1^t (t-1)s - e^{-t}(s^2-s)e^s\,\mathrm{d}s = t^3 - 3t^2 + 5t - 5 + 2e^{1-t}\,.$$